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THE

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ὅ τι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

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PURE AND APPLIED MATHEMATICS.

NOTES ON THE HIGHER ALGEBRA.

By JAMES COCKLE.

(Concluded from Vol. IV. p. 57.)

I NOW propose to consider certain points in the theory of cyclical functions, of Abelian and other cubics, of the second Eulerian quintic, and of the higher prime equations; and to give an exposition of M. Hermite's argument respecting equations of the fifth degree, with a few observations on that of Sir W. R. Hamilton.

ON CYCLICAL FUNCTIONS.

The theorem which Mr. Harley has been pleased to notice in Art. 3 of his paper "On the Theory of Quintics," (*Quarterly Journal*, Jan., 1860) is but one of a class of similar propositions. Let

$$(\omega, x^r) = x_1^{\,r} + \omega x_2^{\,r} + \omega^2 x_3^{\,r} + \ldots + \omega^{n-1} x_n^{\,r},$$

where ω is an unreal n^{th} root of unity, n being a prime number, and $x_1, x_2, \ldots x_n$ are the roots of

$$fx = 0,$$

an algebraic equation of the n^{th} degree put under the usual form. Then

$$\begin{aligned}(\omega, x^r).(\omega^{n-1}, x^r) &= \Sigma' \{x^r.(\omega^{n-1}, x^r)\}\\ &= \Sigma' \{x^r.(\omega, x^r)\},\end{aligned}$$

where Σ' is the cyclical symbol of Mr. Harley. Hence, adopting the latter form, we have

$$(\omega, x^r).(\omega^{n-1}, x^s) + (\omega^{n-1}, x^r).(\omega, x^s) +$$
$$(\omega^2, x^r).(\omega^{n-2}, x^s) + (\omega^{n-2}, x^r).(\omega^2, x^s) +$$
$$\ldots + (\omega^{\frac{n-1}{2}}, x^r).(\omega^{\frac{n+1}{2}}, x^s) + (\omega^{\frac{n+1}{2}}, x^r).(\omega^{\frac{n-1}{2}}, x^s)$$
$$= \Sigma' [x^s.\{(\omega, x^r) + (\omega^{n-1}, x^r) + (\omega^2, x^r) + (\omega^{n-2}, x^r)$$
$$+ \ldots + (\omega^{\frac{n-1}{2}}, x^r) + (\omega^{\frac{n+1}{2}}, x^r)\}]$$
$$= \Sigma' [x^s \{(\omega, x^r) + (\omega^2, x^r) + (\omega^3, x^r) + \ldots + (\omega^{n-1}, x^r)\}]$$
$$= \Sigma' [x_1^s \{(n-1)\, x_1^r - x_2^r - x_3^r - \ldots - x_n^r\}]$$
$$= (n-1)\, \Sigma' x_1^{r+s} - \Sigma' x_1^s (x_2^r + x_3^r + \ldots + x_n^r)$$
$$= (n-1)\, \Sigma x^{r+s} - \Sigma x_1^r x_2^s.$$

Making $s = r$, we have

$$(\omega, x^r).(\omega^{n-1}, x^r) + (\omega^2, x^r).(\omega^{n-2}, x^r) + \ldots + (\omega^{\frac{n-1}{2}}, x^r).(\omega^{\frac{n+1}{2}}, x^r)$$
$$= \frac{n-1}{2} \Sigma x^{2r} - \Sigma x_1^r x_2^r.$$

From the structure of these functions we perceive that when $s = r = 1$ they are critical, and leading coefficients (or sources) of covariants. They will probably all be found to have a place in the theory of the higher equations. Indeed, in the case of $s = r = 1$ the theory of Euler, which for quintics (see Art. 1 of Mr. Harley's paper) gives

$$(\omega, x).(\omega^4, x) + (\omega^2, x).(\omega^3, x) = 25P,$$

indicates as much. These critical functions, I may add, seem to have their application in the theory of elimination. And if we transform $fx = 0$ to an equation in y,

$$y^n + Ay^{n-1} + By^{n-2} + Cy^{n-3} + \ldots + Uy + V = 0,$$

in which

$$y = P + Qx + Rx^2 + \ldots + Lx^{m-1},$$

the result of the elimination of P between $A = 0$ and $B = 0$ is obtained by simply writing down the critical quadratic function, previously expunging from A and B the terms into which P enters: thus

$$2nB_{P=0} - (n-1)\, A_{P=0} = 0.$$

In like manner the result of the elimination of P between $A=0$ and $C=0$ is given by

$$3n^2C_{P=0}-3n(n-2)B_{P=0}A_{P=0}+(n-1)(n-2)A_{P=0}{}^3=0.$$

The results of the elimination of P between $A=0$ and $D=0$, $B=0$ and $C=0$, &c. require special consideration, inasmuch as the number of distinct critical functions of the fourth and higher degrees is not, as in the case of those of the second and third degrees, one only.

ON ABELIAN AND OTHER CUBICS.

Mr. Jerrard (*Phil. Mag.*, S. 4, Vol. III., pp. 459–460) citing Legendre's *Théorie des Nombres*, 3rd edition, Vol. II., p. 438, adverts to an antagonism between the results, at which Legendre had arrived, and those of Abel. That Mr. Jerrard is right in supposing that Legendre has overlooked the existence of an equation of condition will appear from the following theorem. The point left doubtful by Mr. Jerrard (*ib.* p. 459, note) is here decided: b_2 vanishes, that is to say, the rational function θx does not contain x^2.

In order that a cubic may be an Abelian, it is necessary, and sufficient, that its quadratic critical function should vanish.

Let
$$fx=0,$$
or
$$x^3+ax^2+bx+c=0,$$
be the cubic, and x_1 and x_2 two of its roots. Then

$$(fx_2-fx_1)\div(x_2-x_1)=0,$$

is equivalent to

$$x_2{}^2+(x_1+a)\,x_2+x_1{}^2+ax_1+b=0,$$

which, solved as a quadratic, gives

$$x_2=-\frac{a+x_1}{2}\pm\sqrt{\left(\frac{a^2-4b-2ax_1-3x_1{}^2}{4}\right)}.$$

Now in order that the expression under the radical may be rational, we must have

$$-3(a^2-4b)=a^2,$$
or
$$a^2-3b=0,$$
in which case

$$x_2=-\frac{a+x_1}{2}\pm\frac{1}{\sqrt{(-12)}}(a+3x_1).$$

The theorem, thus proved, may be verified* as follows. Write

$$x_2 = -\frac{a+x_1}{2} + \frac{1}{\sqrt{(-12)}}(a+3x_1),$$

$$x_3 = -\frac{a+x_1}{2} - \frac{1}{\sqrt{(-12)}}(a+3x_1),$$

relations which may be expressed by

$$x' = \theta x_1 = -\frac{a+x_1}{2} \pm \frac{1}{\sqrt{(-12)}}(a+3x_1).$$

Then

$$\theta^2 x_1 = \theta\theta x_1 = \theta x' = -\frac{a+x'}{2} \pm \frac{1}{\sqrt{(-12)}}(a+3x')$$

$$= -\frac{a}{2} + \frac{a}{4} + \frac{x_1}{4} \mp \frac{1}{\sqrt{(-12)}}\left(\frac{a}{2} + \frac{3x_1}{2}\right)$$

$$\pm \frac{1}{\sqrt{(-12)}}\left\{a - \frac{3a}{2} - \frac{3x_1}{2} \pm \frac{3}{\sqrt{(-12)}}(a+3x_1)\right\}$$

$$= -\frac{a}{2} - \frac{x_1}{2} \mp \frac{1}{\sqrt{(-12)}}(a+3x_1),$$

and $\theta^3 x_1 = \theta^2\theta x_1 = \theta^2 x'$

$$= -\frac{a}{2} - \frac{x'}{2} \mp \frac{1}{\sqrt{(-12)}}(a+3x')$$

$$= -\frac{a}{2} + \frac{a}{4} + \frac{x_1}{4} \mp \frac{1}{\sqrt{(-12)}}\left(\frac{a}{2} + \frac{3x_1}{2}\right)$$

$$\mp \frac{1}{\sqrt{(-12)}}\left\{a - \frac{3a}{2} - \frac{3x_1}{2} \pm \frac{3}{\sqrt{(-12)}}(a+3x)\right\}$$

$$= x_1.$$

Consequently the cubic is an Abelian. If in this cubic we substitute x' for x_1, we are led, after reduction, to a result of the form

$$fx' = Fx_1 fx_1 = 0,$$

and not to one of the form

$$fx' = 0 + 0x_1 + 0x_1^2 + \ldots + 0x_1^r.$$

* This verification (which shows that the cubic possesses all the properties of an Abelian) is, as will presently be seen, material for ulterior purposes. The Abelian cubic, I may add, is of the form

$$\left(x + \frac{a}{3}\right)^3 + c - \frac{a^3}{27} = 0.$$

The latter form would, indeed, indicate a spurious or illusory solution, such as would be given by

$$x = \sqrt[3]{(-ax^2 - bx - c)},$$

a relation which, unlike

$$x = \theta x, \text{ or } x - \theta x = 0,$$

is identical in substance with $fx = 0$.

In the present case the form of F may perhaps be best determined by considering the general cubic in which, if we make

$$X = a + x_1,$$

we arrive at

$$x_2 = -\frac{X}{2} + \sqrt{\left(\frac{X^2}{4} + \frac{c}{x_1}\right)}.$$

Substituting this value in $fx_2 = 0$, and reducing, we find that radicals disappear, and that

$$fx_2 = fx_1,$$

and, consequently, $Fx_1 = 1$.

In the general cubic let

$$R(x) = \frac{a^2}{4} - b - \frac{ax}{2} - \frac{3x^2}{4},$$

and, as before,

$$x' = -\frac{a}{2} - \frac{x}{2} + \sqrt{\{R(x)\}} = \phi x,$$

then
$$\phi^2 x = \phi\phi x = \phi x'$$
$$= -\frac{a}{2} - \frac{x'}{2} + \sqrt{\{R(x')\}}$$
$$= -\frac{a}{2} + \frac{a}{4} + \frac{x}{4} - \frac{\sqrt{\{R(x)\}}}{2}$$
$$\pm \sqrt{\left\{\left(\frac{a}{4} + \frac{3x}{4}\right)^2 - \sqrt{R(x)}.\left(-\frac{a}{4} - \frac{3x}{4}\right) + \frac{R(x)}{4}\right\}}$$
$$= -\frac{a}{4} + \frac{x}{4} - \frac{\sqrt{\{R(x)\}}}{2} \pm \left[\frac{a}{4} + \frac{3x}{4} + \frac{\sqrt{\{R(x)\}}}{2}\right].$$

Consequently, either

$$\phi^2 x = x,$$

or
$$\phi^2 x = -\frac{a}{2} - \frac{x}{2} - \frac{\sqrt{\{R(x)\}}}{2}.$$

But, inasmuch as the middle term of the square of which the root has been extracted involves $(-1)^2$, the negative sign is to be taken. Hence, proceeding as before,

$$\phi^3 x = \phi^2 \phi x = \phi^2 x'$$

$$= -\frac{a}{2} - \frac{x'}{2} - \sqrt{\{R(x')\}}$$

$$= -\frac{a}{4} + \frac{x}{4} - \frac{\sqrt{\{R(x)\}}}{2} \mp \left[\frac{a}{4} + \frac{3x}{4} + \frac{\sqrt{\{R(x)\}}}{2}\right].$$

Consequently, either

$$\phi^3 x = -\frac{a}{2} - \frac{x}{2} - \sqrt{\{R(x)\}} = \phi^2 x,$$

or

$$\phi^3 x = x.$$

The latter is the true result, for $-\sqrt{\{(-1)\}^2}$ is positive. Or, without having recourse to an *a priori* consideration of the signs, a reference to the results obtained in the case of Abelian cubics, indicates which value of $\phi^{(r)}$ is to be taken, and shows that the roots of a general cubic may be represented by

$$x,\ \phi x,\ \phi^2 x,$$

where $\phi^3 x = x$, and ϕ is irrational. Moreover,

$$x - \phi x$$

does not vanish, and consequently the solutions are not illusory.*

ON THE SECOND EULERIAN QUINTIC.

The general theory of Euler, so far as it relates to quintics, is sufficiently explained in Mr. Harley's paper in the January (1860) number. I here propose to give a brief discussion of the second solvible quintic arrived at by Euler, in Vol. IX. of the Petersburgh *Novi Commentarii.* And I shall introduce the discussion by giving the words of Euler as they appear at pp. 96–98 of the volume just mentioned. This mode of introducing the subject will be welcome to those who have not convenient access to the original, and it is the more expedient inasmuch as the form in question is but little known, and Meyer Hirsch does

* Interesting matter relating to cubics will be found at pp. XXIV.–XXXI. of the Introduction to Barlow's Tables (1814). Some of Mr. Lockhart's researches on cubics (see *Mathematician*, Vol. I. p. 334; II. p. 42) in the year 1845, which seem to have attracted the favourable notice of Professor Hearn (ib. Vol. II. p. 43), are connected with the subject of the *Seizième Leçon* of M. Serret's *Cours*.

not (see pp. 201–4 of Ross's translation of his "Collection" &c.) give it. It seems to belong to a special class of soluble quintics. Euler's words and formulæ are as follows:—

"44. Possunt autem ex forma generali innumerabiles deduci æquationes quinti ordinis, quarum radices assignari licet, etiamsi ipsæ illæ æquationes in factores resolui nequeant. Proposita enim æquatione quinti gradus:

$$x^5 = Ax^3 + Bx^2 + Cx + D,$$

cuius coefficientes habeant sequentes valores: $A = \frac{5}{gk}(g^3 + k^3)$,

$$B = \frac{5}{2mnrr}\{(m+n)(m^2g^3 - n^2k^3) - (m-n)\,rr\},$$

$$C = \frac{5}{mnggkkrr}[g^3(m^2g^3 - n^2k^3)^2 - \{m(m+n)\,g^6 - (m^2 + mn - n^2)\,g^3k^3 + n(m-n)\,k^6\}\,rr - k^3r^4],$$

$$\begin{aligned} D = {} & \frac{gg}{mmnk^4g^3}\{(m^2g^3 - nnk^3)^3 - (m^2g^3 - n^2k^3)(m^2g^3 + rn^2k^3) - n^2k^3r^4\} \\ & + \frac{kk}{mnng^4r^3}\{m^2g^3r^2(m^2g^3 - n^2k^3) - (2m^2g^3 + n^2k^3)\,r^4 + r^6\} \\ & + \frac{5(m-n)(g^3 - k^3)(m^2g^3 - n^2k^3)}{2mngkrr} - \frac{5(m+n)(g^3k^3)}{2mngk}, \end{aligned}$$

eius radices semper assignari possunt.

"45. Ponatur enim breuitatis gratia:

$$T = (m^2g^3 - n^2k^3)^2 - 2(m^2g^3 + n^2k^3)\,rr + r^4,$$

sitque: $$\left.\begin{matrix}P\\Q\end{matrix}\right\} = \frac{(m^2g^3 - n^2k^3)^3 - (m^2g^3 - n^2k^3)(m^2g^3 + 2n^2k^3)\,rr - n^2k^3r^4 \pm \{(m^2g^3 - n^2k^3)^2 - n^2k^3rr\}\sqrt{(T)}}{2mnnr},$$

$$\left.\begin{array}{l}R\\S\end{array}\right\}=\frac{(m^2g^3-n^2k^3)\,m^2g^3-(2m^2g^3+n^2k^3)\,rr+r^4\pm(m^2g^3-rr)\sqrt{(T)}}{2mnnr},$$

vbi signa superiora pro valoribus P et R, inferiora pro Q et S valent, ac quælibet radix æquationis erit:

$$x=\mathfrak{a}\sqrt[5]{\frac{gg}{k^4}}\,P+\mathfrak{a}^2\sqrt[5]{\frac{kk}{g^4}}\,R+\mathfrak{a}^3\sqrt[5]{\frac{kk}{g^4}}\,S+\mathfrak{a}^4\sqrt[5]{\frac{gg}{k^4}}\,Q.$$

"46. Vt rem exemplis illustremus, ex his formis sequentia formari possunt:

"I. Æquationis $x^5=40x^3+70xx-50x-98$, radix est

$$x=\sqrt[5]{\{-31-3\sqrt{(-7)}\}}+\sqrt[5]{(-18+10\sqrt{-7})}+\sqrt[5]{(-18-\sqrt{-7})}$$
$$+\sqrt[5]{(-31-3\sqrt{-7})}.$$

"II. Æquationis $x^5=2625x+16600$, radix est

$$x=\sqrt[5]{75}\,(5+4\sqrt{10})+\sqrt[5]{225}\,(35+11\sqrt{10})$$
$$+\sqrt[5]{225}\,(35-11\sqrt{10})+\sqrt[5]{75}\,(5-4\sqrt{10}),$$

quæ eo magis sunt notatu digna, quod hæ æquationes nullo alio modo resolui possunt. Simili autem modo huiusmodi inuestigationes ad æquationes altiorum graduum extendi possunt: facileque erit, ex quouis gradu innumerabiles æquationes per alias methodos irresolubiles exhibere, quarum huius methodi ope non solum vna, sed omnes plane radices exhiberi queant."

Now, as I have elsewhere (*Phil. Mag.*, May, 1858, p. 389) pointed out, these formulæ will be found to give the results

$$PQ=\left(\frac{m}{n}\right)^2 r^4k^3g^6,\quad RS=g^3k^6,$$

and (by comparison with other formulæ of Euler's)

$$\sqrt[5]{\left(\frac{mr^2}{n}\right)^2}=1,$$

which is, I think, a necessary limitation.

Hence, recurring to Mr. Harley's paper "On the Theory of Quintics," and to mine "On the Resolution of Quintics," (see the January, 1860, and June, 1860, Nos. of this *Journal* respectively) we see that

$$\beta_1\beta_4=\sqrt[5]{\frac{g^4}{k^8}}\,k^3g^6=\frac{g^3}{k},\quad \beta_2\beta_3=\sqrt[5]{\frac{k^4}{g^8}}\,g^3k^6=\frac{k^3}{g}.$$

Hence, again, the equation in ϑ (by which symbol I designate the function

$$\tfrac{1}{25} f\omega f\omega^4),$$

will be

$$\vartheta^2 - \frac{A}{5}\vartheta + gk = 0,$$

or

$$\vartheta^2 - \left(\frac{g^2}{k} + \frac{k^2}{g}\right)\vartheta + gk = 0,$$

and one of the values of the "resolvent product" will be

$$\theta = 5^4 gk.$$

The foregoing result affords some verification of Euler's formulæ which, coming from so great a master, deserve a careful consideration. Euler does not employ a resolvent sextic, but if we view his second soluble form in the light of the theories which lead to such a sextic, I am inclined to think that it will be found to appertain to quintics of which the resolvent sextic has one rational root. In order that a quintic may be soluble its resolvent must be either (1) reducible or (2) Abelian. If reducible it may be decomposable into a linear and a quintic factor. Perhaps such decomposition, as distinguished from that into quadratic or cubic factors, characterizes Euler's second soluble form, as well as the soluble quintic of Abel.

ON THE HIGHER PRIME EQUATIONS.

To *solve by radicals any general equation higher than a biquadratic* is an impossible problem. But if, with M. Hermite, we extend the term solution to results in which the roots of a given equation are represented by uniform functions, we may seek, by introducing auxiliary variables, to express the roots of any equation, separately, by as many distinct and uniform functions of the new variables, as there are distinct roots of the equation. See pp. 1–2 of M. Hermite's Essay *Sur la théorie des Equations Modulaires et la Résolution de l'Equation du Cinquième Degré.* Paris, 1859.

In discussing the properties of such functions, we may start with the proposition that every function of them of the form

$$x_1^m + x_2^m + \ldots + x_n^m,$$

is, when m is an integer, capable of being expressed as a rational function of the coefficients of the given equation.

But we cannot, in dealing with the uniform functions separately, either adopt, or assume the adoption of, that process of rationalization, the abstract possibility of which has formed the basis of the propositions of Abel.* This circumstance constitutes the characteristic difference between the theories of the higher and lower equations.*

Could we isolate and determine any one root of a general n-ic equation, that equation, as is well known, could be depressed to the $(n-1)^{\text{th}}$ degree.

Let (x) denote an (in general transcendental or non-algebraic) expression for a root of an n-ic equation. The form of (x) is in general unknown, but the hypothesis of its existence enables us to arrive at some general propositions respecting the uniform functions (if any), by which all the roots may be sought to be represented. And particular results, which have been obtained in the case of quintics, will be seen to flow from these general propositions. Thus, forming the function

$$\frac{fx - f(x)}{x - (x)},$$

and denoting it by $F\{x, (x)\}$, we at once perceive that all the roots of $fx = 0$, excepting (x), are roots of the general† equation of the $(n-1)^{\text{th}}$ degree

$$F\{x, (x)\} = 0.$$

* Let $$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
denote any general equation, and let x', an algebraic function of $p_1, p_2, \ldots p_n$, represent one of its roots. By giving to every radical in x' every value of which it is susceptible (*i.e.*, applying to it different unreal roots of unity), multiplying together all the values of $x - x'$ so obtained, and equating the product to zero, we are conducted to a result of the form
$$x^s + P_1 x^{s-1} + P_2 x^{s-2} + \ldots + P_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
in which $P_1, P_2, \ldots P_n$ are rational functions of $p_1, p_2, \ldots p_n$, and in which, too, s cannot be less than n, otherwise (1) would not be irreducible, and, consequently, not general.

From the irreducibility of (1) it follows that if $s = n$, all, and if s be greater than n, some n, of the values of x' will satisfy it, or, as Abel expresses it, that if an equation can be satisfied algebraically it can be solved algebraically. It follows moreover, according to Abel and Sir W. R. Hamilton, that if in x' all composite surds, or surds with composite indices, be resolved into prime surds [*ex. gr.* if $\sqrt[4]{(a)}$ be written $\sqrt{\{\sqrt{(a)}\}}$], and all reducible prime surds be expelled, *all* the values of x' are common to (1) and (2), and, consequently, that $s = n$.

† Only one of the $n+1$ coefficients of the original equation will be wanting in the reduced equation. If $x = \psi a$ be a solution of a symmetric equation

$$F(x, a) = 0,$$

Hence the uniform functions (if any), which represent the n roots of an n-ic equation, are each of them capable of being expressed in terms of one of them, and of the roots of a general equation of the $(n-1)^{\text{th}}$ degree. And, if $n=5$, four of the five uniform functions which, in this case, we know to exist, must be capable of expression as algebraic functions of the fifth. Compare pp. 1–8 of M. Hermite's Essay.

In the preceding title but one of these notes I have shown that all the roots of any cubic may be represented by the series of expressions

$$x,\ \phi x,\ \phi^2 x,$$

where x is one of its roots and ϕ not, in general, a rational function. Let ϕx (a function which, for reasons also given under the preceding title, must not satisfy the relation

$$x-\phi x=0),$$

satisfy

$$f\phi x=0.$$

Make $\phi x=x_2$, then, $fx=0$ being general and consequently irreducible, the equation $fx_2=0$ leads to

$$f\phi x_2=0,$$

and this again to

$$f\phi\phi x=f\phi^2 x=0.$$

Thus, continuing the process, we are led to conclude that all the functions of the series

$$x,\ \phi x,\ \phi^2 x,\ \ldots\phi^r x\ldots,$$

will be roots of $fx=0$, and that, when n is prime, each of the functions

$$x,\ \phi x,\ \phi^2 x,\ \ldots\phi^{n-1} x,$$

will represent a different root of $fx=0$.

$a=\psi x$ will also be a solution, and the roots may be so selected that an equation $\psi^2 a=a$ will hold for one of the forms of ψ. Thus if

$$x^n+a^n=1,$$

we have for the arithmetical root

$$x=\sqrt[n]{(1-a^n)}=\psi a,$$

$$\psi^2 a=\sqrt[n]{[1-\{\sqrt[n]{(1-a^n)}\}^n]}=a.$$

But this does not hold for all the roots, which are contained in the series

$$\psi a,\ \omega\psi a,\ \omega^2\psi a,\ \ldots\omega^{n-1}\psi a.$$

Thus we may have, if $x=\psi' a=\omega\psi a$,

$$\psi'^2 a=\omega\sqrt[n]{\{1-\omega^n(1-a^n)\}}=\omega^2\psi a=\omega\psi' a.$$

For, suppose that the series terminates with $\phi^{m-1}x$, so that $\phi^m x = x$, m being less than n, and

$$n = qm + r;$$

then the roots of $fx = 0$ may be distributed into q groups of m roots

$$\begin{array}{lll} x_1, & \phi x_1, & \ldots \phi^{m-1}x_1, \\ x_2, & \phi x_2, & \ldots \phi^{m-1}x_2, \\ \vdots & \vdots & \quad\vdots \\ x_q, & \phi x_q, & \ldots \phi^{m-1}x_q, \end{array}$$

and a group of r roots

$$x_k, \ \phi x_k, \ \ldots \phi^r x_k,$$

which, provided that x_2 does not form part of the series in x_1, nor x_3 of those in x_1 and x_2 ..., nor x_q of those in $x_1, x_2, \ldots x_{q-1}$, nor x_k of any of the preceding series, will exhaust* the roots of $fx = 0$. But we also know that

$$\phi^{r+1}x_k, \ \phi^{r+2}x_k, \ \ldots \phi^{m-1}x_k, \ \ldots,$$

are roots of $fx = 0$. Hence, either

$$\phi^{r+1}x_k = \phi^{r-s}x_k,$$

* Thus if x_2 does not enter into the series for x_1, neither can $\phi^r x_2$ enter into that series, otherwise we should have

$$\phi^r x_2 = \phi^s x_1, \text{ or } x_2 = \phi^{s-r} x_1,$$

and x_2 would thus form part of the x_1 series.

And in general, if x_r forms no part of the x_{r-p} series, neither can $\theta^s x_r$ form any part of that series. The argument in the text which is, to a certain extent, analogous to that used in the case of Abelian equations (see Serret, *Cours*, 1854, *Leçon* 27), may in the case of a cubic be illustrated thus: let x_1 and ϕx_1 each satisfy the cubic, and let ϕx_1 equal, say x_2, and let the series, if possible, terminate with ϕx_1, so that in fact $\phi^2 x_1 = x_1$. The three expressions

$$x_1, \ \phi x_1, \ x_3,$$

will be roots of the cubic. So also will ϕx_3, which must be equal either to x_1 or to ϕx_1, otherwise $\phi x_3 = x_3$ would lead us to $\phi x_1 = x_1$, which would be inconsistent with the irreducibility of the cubic. But if $\phi x_3 = \phi x_1$ we are led to the equally objectionable result $x_3 = x_1$. The only remaining relation is $\phi x_3 = x_1$, or $x_3 = \phi x_1 = x_2$, which is also inconsistent with the irreducibility of the cubic. Consequently the series cannot terminate with ϕx_1. Nor can it continue, without recurrence, beyond $\phi^2 x_1$: for $\phi^3 x_1$, if not equal to x_1, is equal either to ϕx_1 or $\phi^2 x_1$. But from these equations we deduce, respectively,

$$\phi^2 x_1 = x_1, \text{ or } \phi x_1 = x_1,$$

either of which is inconsistent with the irreducibility of the cubic.

or
$$\phi^{r+1}x_k = \phi^s x_p,$$
where p differs from k, and s is not greater than r.

From the first of these conditions we deduce
$$\phi^{s+1}x_k = x_k,$$
and, giving s its maximum value r, we see that each of the series terminates at the r^{th} term at latest. But this is inconsistent with the supposition that they do not terminate before the m^{th} term, m being greater than r.

Secondly, let
$$\phi^{r+1}x_k = \phi^t x_p,$$
or
$$x_k = \phi^{t-r-1}x_p = \phi^{m+t-r-1}x_p.$$
This relation is inconsistent with the fact that x_k forms no part of any of the preceding series. Consequently, when n is prime, each of the functions x, ϕx, $\ldots\phi^{n-1}x$ will represent a different root of $fx=0$.

Lastly, $\phi^n x = x$. For, if not, let
$$\phi^n x = \phi^{n-r}x,$$
then $\phi^r x = x$, or the series terminates before the n^{th} term, contrary to what has just been proved.

It follows that, if (x) and $\phi(x)$ represent two of the roots of a general n-ic equation, all the roots will, when n is prime, be given by the series
$$(x),\ \phi(x),\ \phi^2(x),\ \ldots\phi^{n-1}(x),$$
which recurs after the n^{th} term.

These functions are uniform in one sense, but not in that in which the term has been used in what precedes. Let
$$(x) = \chi(I, J, \ldots, L),$$
where $I, J, \ldots, L$ are functions of the coefficients of the given equation. Then
$$\phi(x) = \phi\chi(I, J, \ldots, L),$$
and the condition requisite in order that the functions may be uniform, in the wider sense, is
$$\phi\chi(I, J, \ldots, L) = \chi(I_1, J_1, \ldots, L_1),$$
where $I_1, J_1, \ldots, L_1$ are functions of the same description as $I, J, \ldots, L$. But, if the last relation be satisfied, we have also
$$\phi^2\chi(I, J, \ldots, L) = \phi\chi(I_1, J_1, \ldots, L_1) = \chi(I_2, J_2, \ldots, L_2),$$

where $I_2, J_2, \ldots, L_2$ are the same functions of $I_1, J_1, \ldots, L_1$ that $I_1, J_1, \ldots, L_1$ are of $I, J, \ldots, L$. And, continuing the process, we shall be led to

$$\chi(I, J, \ldots, L), \chi(I_1, J_1, \ldots, L_1), \ldots, \chi(I_{n-1}, J_{n-1}, \ldots, L_{n-1}),$$

as the expressions for the n roots. And, since $\phi^n(x) = (x)$, this series will recur after the n^{th} term.

Hence, if two of the roots of a general equation of a prime degree can be represented by uniform functions, all can be so represented: the form of ϕ, which depends upon the solution of the equation of the $(n-1)^{\text{th}}$ degree, being treated as known. If

$$I_1 = I + i,\ J_1 = J + j,\ \ldots,\ L_1 = L + l,$$

we perceive that

$$I_r = I + ri,\ J_r = J + rj,\ \ldots,\ L_r = L + rl,$$

or if

$$I_1 = iI,\ J_1 = jJ,\ \ldots,\ L_1 = lL,$$

we perceive that

$$I_r = i^r I,\ J_1 = j^r J,\ \ldots,\ L_r = l^r L.$$

Each of these forms (which may be, and possibly are, combined in certain cases) is illustrated in the solution of a quintic by elliptic functions. For if λ, μ, and ν be functional symbols, and

$$i = 16,\ j = 0,\ k = 0,$$

we know (see p. 7 of M. Hermite's Essay) that

$$x_r = \frac{\lambda\{I + (r-1)\,i\}}{\mu J.\nu K};$$

and it may be added that the form in $\sqrt[5]{(q)}$ (ibid), expressed in the notation of this paper, may be written

$$x_r = \frac{\lambda\,(i^{r-1} I)}{\mu J.\nu K},$$

provided i be an unreal fifth root of unity, and

$$j = 1,\ k = 1.$$

The general quintic involves, in ultimate analysis, only one parameter; the general sextic two. No general equation, prime or other, higher than a quintic, has yet been solved, and two distinct questions seem to arise respecting their solution: (1) Can some one root be expressed by some definite non-algebraic function? (2) Can all the roots be expressed by a system of uniform functions? It will be borne in mind that the important equations termed "modular"

are not general. Their analogues in the algebraic theory are (as the form of their solutions show), not Abelian equations, but reducible equations with (only) one rational root. The solution of general equations appears to lead to the discussion of differential relations of the class

$$\frac{d\phi x}{dx} = \mathrm{F}\phi \mathrm{f} x,$$

where the forms of F and f are known, and that of ϕ unknown. In the cases of quadratics and cubics I have obtained solutions by the Integral Calculus. But as to this mode of solution I must, for the present at all events, content myself with a reference to papers published elsewhere.* In the theory of cubics f is periodic and $\mathrm{f}^2 x = x$. There are other periodic forms which render the integration easy, when F is suitably assigned.

ON M. HERMITE'S ARGUMENT.

M. Hermite's argument may help to settle a still vexed question. It is as follows:

Let us assume that between the roots of the sextic *réduite* of the general quintic there exist relations which render that sextic an Abelian. These relations would, in effect lead to the conclusion that the *réduite* is resolvible algebraically by quadratic and cubic radicals, and, without having recourse to the demonstration of Abel, we may at once convince ourselves that it would follow that the equation of the fifth degree is resolvible by radicals of the same kind. Let us call x_0, x_1, x_2, x_3, x_4 the roots of this equation, and put

$$u = x_0x_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_0,$$

$$v = x_0x_2 + x_2x_4 + x_4x_1 + x_1x_3 + x_3x_0,$$

the quantities $u+v$ and uv, will be, the one rational and the other a root of an equation of the sixth degree resolvible algebraically by hypothesis. Then u and v and their various values will be expressed by means of quadratic and cubic radicals. The same conclusion will hold with respect to the more general functions

$$u_\alpha = (x_0x_1)^\alpha + (x_1x_2)^\alpha + (x_2x_3)^\alpha + (x_3x_4)^\alpha + (x_4x_0)^\alpha,$$

$$v_\alpha = (x_0x_2)^\alpha + (x_2x_4)^\alpha + (x_4x_1)^\alpha + (x_1x_3)^\alpha + (x_3x_0)^\alpha,$$

* See the *Philosophical Magazine* for August, and November, 1860; and for February, 1861. [Feb. 21, 1861.—J. C.]

whatever be the integral exponent a. It follows that

$$x_0x_1,\ x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_0,$$

for example, will satisfy an equation of the fifth degree, the coefficients of which will only involve radicals of the kind in question. But *with two values* of u_a and v_a it will be possible to form two equations of the fifth degree having *a common root*, for example x_0x_1, the others being different. Hence we may deduce x_0x_1, and consequently the similar function x_0+x_1, in terms of cubic and quadratic radicals, consequently also x_0 and x_1, so that the equation of the fifth degree would be resolvible without quintic radicals.

Such a conclusion is of course inadmissible. M. Hermite's argument is given, with developments, in his "Considerations sur la Résolution Algébrique de l'équation du 5° degré." See pp. 326–336 of Vol. I. of the *Nouvelles Annales de Mathématiques* (par M. M. Terquem et Gerono), 1842.

ON SIR W. R. HAMILTON'S ARGUMENT.

Sir William Rowan Hamilton's disquisition "On the Argument of Abel, respecting the Impossibility of expressing a Root of any General Equation above the Fourth Degree, by any finite Combination of Radicals and Rational Functions" is printed at pp. 171–259 of Vol. XVIII. (Part II., 1839) of *The Transactions of the Royal Irish Academy*. Although the whole of this paper was suggested by the Argument of Abel, and may be said to be a commentary thereon; yet (p. 248) there are considerable differences between the one method of proof and the other. More particularly, in establishing the cardinal proposition that every radical in every irreducible expression for any one of the roots of any general equation is a rational function of those roots, it appeared to Sir W. R. Hamilton more satisfactory to begin by showing that the radicals of highest order will have that property, if those of lower orders have it, descending thus to radicals of the lowest order, and afterwards ascending again; than to attempt, as Abel has done, to prove the theorem, in the first instance, for radicals of the highest order. In fact, while following this last-mentioned method, Abel has been led to assume that the first power of some highest radical can always be rendered equal to unity, by introducing (generally) a new radical, $\sqrt[n]{(pq_1^{n})}$ instead of $\sqrt[n]{(p)}$ for example. But although the quantity under the radical sign is now free from that irrationality of the (say) m^{th}

order which was introduced by the radical $\sqrt[m]{(p)}$, it is not, in general, free from the irrationalities of the same order introduced by the other radicals of that order; and consequently the new radical, to which this process conducts, is in general elevated to the order $m+1$; a circumstance which Abel does not seem to have remarked, and which renders it difficult to judge of the validity of his subsequent reasoning. The opinion of mathematicians not being entirely agreed respecting the possibility or impossibility of expressing a root of the general equation of the fifth degree as a finite combination of radicals and rational functions, Sir W. R. Hamilton's chief object was to assist in deciding opinions upon this important question by developing and illustrating (with alterations) the admirable argument of Abel against the possibility of any such expression for a root of the general equation of the fifth or any higher degree; and by applying the principles of the same argument, to show that no expression of the same kind exists for the root of any general but lower equation, (quadratic, cubic, or biquadratic) essentially distinct from those which have long been known (pp. 175–6).

Wantzel has given a simple proof that no prime power of any unsymmetric function, except its square, is symmetric; that the first radical in the order of calculation is, consequently, a square root; that the roots of any other radicals which enter into the expression for the root of a general equation, higher than a biquadratic, are functions invariable for cyclical permutations of three symbols: and consequently that such an equation is algebraically insoluble (see Serret, *Cours*, 2me. ed., pp. 305–9). I believe that I was wrong in supposing that Wantzel had improperly treated an equation* as an identity, and that his argument was inconclusive. That argument abridges Arts. (21) and (22) of Sir W. R. Hamilton's paper, but does not affect any portion anterior to Art. (13); nor does it, in its present form, supersede Arts. (13) to (20).

4, Pump Court, Temple, London.
September 3, 1860.

* In Wantzel's equation, as in the functional equation

$$\psi(x) = c\psi\left(\frac{1}{x}\right),$$

the first step discloses that the constant must satisfy the relation $c^2 = 1$.

ON SOME APPLICATIONS OF ALGEBRA TO THE THEORY OF COVARIANTS.

By Michael Roberts.

Let $(A_0, A_1, A_2, \ldots A_p)(x, y)^p$, $(B_0, B_1, B_2, \ldots B_q)(x, y)^q$ be two covariants of the quantic

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n.$$

These covariants we shall designate by A and B, and the quantic itself, as is usual, by U. When $y = 1$, let $\alpha_1, \alpha_2, \ldots \alpha_p$ be the roots of $A = 0$, and let $\beta_1, \beta_2, \ldots \beta_q$ be the roots of $B = 0$. Also, let $x_1, x_2, \ldots x_n$ be the roots of $U = 0$, for $y = 1$: and let δ denote the operation

$$a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \ldots + na_{n-1} \frac{d}{da_n}$$

applied to a function of the a's. Since

$$A_0 \alpha_r^p + pA_1 \alpha_r^{p-1} + \frac{p.p-1}{1.2} A_2 \alpha_r^{p-2} + \ldots + A_p = 0,$$

we find

$$p(A_0 \alpha_r^{p-1} + (p-1) A_1 \alpha_r^{p-2} + \ldots + A_{p-1})(1 + \delta\alpha_r) = 0,$$

which gives $$\delta\alpha_r = -1,$$

and in consequence of a remarkable equation given by M. Brioschi, we deduce

$$\left(\frac{d}{dx_1} + \frac{d}{dx_2} + \ldots + \frac{d}{dx_n}\right) \alpha_r = 1,$$

so that $\alpha_r - x_s$ is a function of the differences of the roots $x_1, x_2, \ldots x_n$. Again, let $\beta_{r'}$ be any root of $B = 0$ and

$$\delta(\alpha_r - \beta_{r'}) = 0,$$

so that $\alpha_r - \beta_{r'}$ is a function of the differences of $x_1, x_2, \ldots x_n$: and because A_0, B_0 are functions of the same differences, it follows that the eliminant of A and B is a function of the differences of the roots $x_1, x_2, \ldots x_n$. Now let ϖ be the degree of A_0 in the coefficients of U: and let κ be its degree in the roots $x_1, x_2, \ldots x_n$: and let ϖ' κ be the corresponding

quantities for B_0: hence, as the eliminant of A and B is of the q^{th} degree in the coefficients of A, and of the p^{th} degree in the coefficients of B, its degree in the coefficients of the quantic U is $\varpi q + p\varpi'$: and its degree in $x_1, x_2, \ldots x_n$ is

$$p\kappa' + q\kappa + pq.$$

It is therefore the source of a covariant of U of the degree in the variables

$$n(\varpi q + p\varpi') - 2p\kappa' - 2q\kappa - 2pq.$$

But $$n\varpi - 2\kappa = p,\ n\varpi' - 2\kappa' = q;$$

and these equations reduce to zero the value of the above expression. Hence we derive the following theorem: *The eliminant of two covariants of a quantic is an invariant of the quantic itself.*

In general, let A_0 be a function of the differences of the roots of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0,$$

of the degree ϖ in the a's, and of the degree κ in the roots: and let B_0 be a function of the differences of the roots of the degree ϖ' in the coefficients, and of the degree κ' in the roots. Now suppose

$$(A_0, A_1, A_2, \ldots A_{m\varpi-2\kappa})(x, y)^{m\varpi-2\kappa} \ldots\ldots\ldots (A),$$

to be a covariant of

$$(a_0, a_1, a_2, \ldots a_m)(x, y)^m,$$

and let

$$(B_0, B_1, B_2, \ldots B_{p\varpi'-2\kappa'})(x, y)^{p\varpi'-2\kappa'} \ldots\ldots\ldots (B),$$

be a covariant of

$$(a_0, a_1, a_2, \ldots a_p)(x, y)^p:$$

then the eliminant of (A) and (B) will be the source of a covariant of the primitive form

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n,$$

of the degree $\quad \varpi(p\varpi' - 2\kappa') + \varpi'(m\varpi - 2\kappa)$

in the a's, and of the degree

$$(n-m)\varpi(p\varpi' - 2\kappa') + (n-p)\varpi'(m\varpi - 2\kappa)$$

in the variables. We suppose $n > m$, $n > p$.

For the covariant whose source is the eliminant of

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n,$$

and its $(n-p)^{\text{th}}$ differential coefficient with respect to x, let $m=n$, $\varpi=1$, $\kappa=0$, $\varpi'=1$, $\kappa'=0$, in the above formulæ, and the eliminant gives rise to a covariant of the degree $n+p$ in the a's, and of the degree $n(n-p)$ in the variables. But in this case a_0 is a factor of the eliminant, so that the eliminant (proper) is of the degree $n+p-1$ in the a's, and gives rise to a covariant of the degree $n(n-p-1)$ in the variables. If $p=n-1$, the eliminant, as is well known, becomes the discriminant.

15, Trinity College, Dublin.
March 6, 1861.

ON THE CRITERIA OF MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

By William Walton, M.A., Trinity College.

Let F represent a function of two independent variables x and y. The function F will have a maximum or minimum value, as was first shewn by Lagrange,* if

$$\frac{dF}{dx}=0 \text{ and } \frac{dF}{dy}=0,$$

provided that

$$\frac{d^2F}{dx^2}\cdot\frac{d^2F}{dy^2} > \left(\frac{d^2F}{dx\,dy}\right)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

O'Brien, in the preface to his Differential Calculus, describes the inequality (1) as "the very insufficient and troublesome criterion usually employed in distinguishing the maxima and minima of functions of two variables." He has accordingly not introduced Lagrange's criterion into his Treatise, but has proposed an entirely different method of discussing the general problem in question. His method is thus described in his own words:

"Let $F(x, y)$ be a function of two independent variables x and y, and let $F(x_1, y)$ be a maximum value of $F(x, y)$, determined on the supposition that y is constant and x alone variable, x_1 of course being some function of y; and again

* *Misc. Taurinensia*, Tom. I. p. 19. *Théorie des Fonctions*, Chapitre XI.

let $F(a, b)$ be a maximum value of $F(x_1, y)$ determined on the supposition that y is variable, a being the value of x_1 when y becomes b: then $F(a, b)$ is a maximum value of $F(x, y)$ when x and y are both supposed to vary in any manner."

The objection to Lagrange's criterion consists in this: when $\frac{d^2F}{dx^2} \cdot \frac{d^2F}{dy^2}$ and $\frac{d^2F}{dxdy}$ both vanish, it becomes necessary to proceed to third and fourth differentials, or frequently to still higher orders, the ordinary criterion (1) being then inapplicable.

The method of O'Brien, although elegant in idea and sometimes useful in practice, is however frequently inapplicable by reason of the complexity of the attendant algebraical operations. In fact, when we equate $\frac{dF}{dx}$ to zero, it is sometimes not possible to obtain thence an expression for x in terms of y, and frequently, when it is possible to do so, the expression for x in terms of y is too complex to be made use of.

The object of this communication to the *Quarterly Journal* is to suggest a modification of the ordinary method, as developed by Lagrange, and to point out a correspondingly modified criterion, which, although not always sufficient, is at any rate frequently conclusive when Lagrange's is inapplicable, and which ordinarily diminishes algebraical labour even when both are available.

Using D to denote total and d partial differentiation, we have, putting $dx = ldr$, $dy = mdr$, l and m being constants,

$$\frac{DF}{dr} = l\frac{dF}{dx} + m\frac{dF}{dy} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In order that F may have a maximum or minimum value, in accordance with the relations between x, y, r, it is sufficient and necessary that $\frac{DF}{dr}$ change sign as x and y pass through their critical values.

Let $$\frac{dF}{dx} = P.P'.Q.U, \quad \frac{dF}{dy} = P.P'.R.V,$$

P, an essentially positive function of x and y, being a common factor of both $\frac{dF}{dx}$ and $\frac{dF}{dy}$. Suppose Q, R, to be other

functions of x and y which do not become zero for values of x and y derived from the simultaneous equations

$$\begin{Bmatrix} U=0 \\ V=0 \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}.$$

Let $u_1 = Q_1 . U$, and $v_1 = R_1 . V$, where Q_1 and R_1 are the values acquired by Q and R, respectively, when x and y are replaced by their values derived from the equations (3), and any subsequent positive common factor of Q and R is rejected.

Put $$\frac{Df_1}{dr} = P' (lu_1 + mv_1) \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}.$$

It is evident, in virtue of the principle of change of signs, that if the values of x, y, derived from (3), make F a maximum or minimum, they will also make f_1 a maximum or minimum, respectively; and conversely. Thus the problem of maxima and minima of F is transferred from the consideration of the equation (2) to that of the equation (4).

Again, let P', a common factor of $\frac{dF}{dx}$ and $\frac{dF}{dy}$, be variable in sign and, for the values of x and y derived from the equations (3), either positive or negative. Instead of $P'u_1$ and $P'v_1$ take u and v respectively, the magnitudes of u, v, being the same respectively as those of u_1, v_1, and their signs the same or opposite, accordingly as P', for the values of x, y, derived from (3), is positive or negative.

Put $$\frac{Df}{dr} = lu + mv.$$

Then, if the values of x, y, derived from (3), make f a maximum or minimum, they will also make f_1 and therefore F a maximum or minimum, and conversely: in fact, as Français* has shewn, the equation $P' = 0$ corresponds, geometrically, to a locus of maxima or minima, not to an absolute maximum or minimum. If P' be zero, under the conditions (3), the values of x and y thence derived need not be considered, as being particular instances of the values of x and y in the equation $P' = 0$.

Differentiating again, considering l, m, constant, we have

$$\frac{D^2f}{dr^2} = l^2 \frac{du}{dx} + lm \left(\frac{du}{dy} + \frac{dv}{dx}\right) + m^2 \frac{dv}{dy}.$$

* *Annales de Gergonne*, Vol. III. p. 132.

Then, f being regarded as a function of one variable r, it will, by the theory of maxima and minima of functions of one variable, be a maximum or minimum, when $\frac{Df}{dr}$ is zero and $\frac{D^2f}{dr^2}$ finite.

Conceive next $\frac{Df}{dr}$ to be zero, whatever be the values of l and m: the sufficient and necessary conditions are that $u=0$, $v=0$. Again, as is easily seen algebraically, $\frac{D^2f}{dr^2}$ will be positive or negative for all values whatever of l and m, provided that

$$4\frac{du}{dx}\frac{dv}{dy} > \left(\frac{du}{dy}+\frac{dv}{dx}\right)^2,$$

and will be positive or negative accordingly as $\frac{du}{dx}$ and $\frac{dv}{dy}$ are both positive or both negative. Thus, as our final conclusion, F will be an absolute maximum or minimum, when

$$u=0,\ v=0,$$

$$4\frac{du}{dx}\frac{dv}{dy} > \left(\frac{du}{dy}+\frac{dv}{dx}\right)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

a maximum when $\frac{du}{dx}$ and $\frac{dv}{dy}$ are both negative, and a minimum when they are both positive.

It is obvious that Lagrange's criterion is included in the criterion (5), that is, when u, v, are simply $\frac{dF}{dx}$, $\frac{dF}{dy}$, unmodified.

In reference to the geometry of surfaces, our course of investigation is, first, to find a point on the surface where the value of z is stationary for infinitesimal changes in the values of x and y: secondly, to determine whether this value of z is a maximum or minimum in any proposed plane section of the surface through the coordinate z: and, thirdly, to ascertain whether, for all such plane sections through this point, this value of z is a maximum or minimum.

In regard to the equations (3), it is important to observe that we may disregard the values of x and y derived from them, if either U or V is incapable of a change of sign as x or y, respectively, varies infinitesimally from its critical

value. The rejection of such a pair of values of x and y corresponds geometrically to getting rid of a point on the surface at which z is not a maximum or minimum in a section parallel to the plane of zx or zy.

I will proceed now to illustrate the preceding investigations by a few examples.

1. To find the maximum or minimum value of

$$F=(x^2+y^2)^{\frac{1}{3}}.$$

We have $\dfrac{dF}{dx}=\dfrac{2x}{3(x^2+y^2)^{\frac{2}{3}}}$, $\dfrac{dF}{dy}=\dfrac{2y}{3(x^2+y^2)^{\frac{2}{3}}}$.

Hence

$$u=x,\quad v=y,$$

$$\frac{du}{dx}=1,\quad \frac{dv}{dy}=1,\quad \frac{du}{dy}=0,\quad \frac{dv}{dx}=0.$$

The criterion (5) is therefore satisfied, and $x=0$, $y=0$, correspond to a minimum value of F.

2. To find the maximum or minimum values of

$$F=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2.(x^2+y^2),$$

where a is supposed to be greater than b.

We have

$$\frac{dF}{dx}=2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).\left(\frac{2x^2+2y^2}{a^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)x,$$

$$\frac{dF}{dy}=2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).\left(\frac{2x^2+2y^2}{b^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)y.$$

Since the common factor $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1$ of $\dfrac{dF}{dx}$ and $\dfrac{dF}{dy}$ is to be rejected, excepting so far as sign is concerned, we have to limit our attention to the three following systems of values:

$$\begin{pmatrix}x=0\\y=0\end{pmatrix},\quad \begin{pmatrix}x=0\\y^2=\frac{1}{3}b^2\end{pmatrix},\quad \begin{pmatrix}y=0\\x^2=\frac{1}{3}a^2\end{pmatrix}.$$

First system. From the expressions for $\dfrac{dF}{dx}$ and $\dfrac{dF}{dy}$, we see that

$$u=x,\quad v=y;$$

whence $$\frac{du}{dx}=1,\quad \frac{dv}{dy}=1,\quad \frac{du}{dy}=0,\quad \frac{dv}{dx}=0.$$

The criterion (5) is therefore satisfied, and the corresponding value of F is a minimum.

Second system. Here we have

$$u=\frac{2x}{a^2}(a^2-b^2),\quad v=-b\sqrt{(3)}\left(\frac{2x^2+3y^2}{b^2}+\frac{x^2}{a^2}-1\right);$$

and therefore, for the particular values of x and y,

$$\frac{du}{dx}=\frac{2}{a^2}(a^2-b^2),\quad \frac{dv}{dy}=-6,\quad \frac{du}{dy}=0,\quad \frac{dv}{dx}=0.$$

The criterion (5) is not satisfied, and therefore the second system must be rejected.

Third system. We may ascertain in the same way that

$$\frac{du}{dx}=-6,\quad \frac{dv}{dy}=-\frac{2}{b^2}(a^2-b^2),\quad \frac{du}{dy}=0,\quad \frac{dv}{dx}=0.$$

The criterion (5) is accordingly satisfied, and the value of F is a maximum.

3. To find the maxima or minima of

$$F=a^2x^2y-2ax^3y+x^4y-2ax^2y^2+2x^3y^2+x^2y^3.$$

We have

$$\frac{dF}{dx}=2xy(x+y-a)(2x+y-a),$$

$$\frac{dF}{dy}=x^2(x+y-a)(x+3y-a).$$

It is unnecessary to attend to the common factors x and $x+y-a$ of the two partial differential coefficients.

The only values of x, y, which can be derived from the zero values of the other factors are included in the two systems

$$\begin{pmatrix}x=a\\y=0\end{pmatrix}\text{ and }\begin{pmatrix}x=\frac{2}{5}a\\y=\frac{1}{5}a\end{pmatrix}.$$

The first of these systems must be rejected, because the values of x and y are particular instances of the values, of x and y in the equation

$$x+y-a=0.$$

In regard to the second system, we have

$$u = -2x - y + a = 0, \quad v = -x - 3y + a = 0,$$

$$\frac{du}{dx} = -2, \quad \frac{dv}{dy} = -3, \quad \frac{du}{dy} = -1, \quad \frac{dv}{dx} = -1.$$

The criterion (5) is therefore satisfied, and the second system therefore renders F a maximum.

4. Let $$F = (ax^2 + by^2)\, e^{-x^2-y^2}.$$

Then $$\frac{dF}{dx} = 2x\,(a - ax^2 - by^2).e^{-x^2-y^2},$$

$$\frac{dF}{dy} = 2y\,(b - ax^2 - by^2).e^{-x^2-y^2}.$$

The only values of x and y which can render F a maximum or a minimum, are given in the three systems

$$\begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix}, \quad \begin{pmatrix} x = 0 \\ y = \pm 1 \end{pmatrix}, \quad \begin{pmatrix} x = \pm 1 \\ y = 0 \end{pmatrix}.$$

Taking the first system, we have

$$u = ax, \quad v = by,$$

$$\frac{du}{dx} = a, \quad \frac{dv}{dy} = b, \quad \frac{du}{dy} = 0, \quad \frac{dv}{dx} = 0,$$

and accordingly F has a minimum value.

Taking the second system,

$$u = (a - b)\,x, \quad v = \pm(b - ax^2 - by^2),$$

$$\frac{du}{dx} = a - b, \quad \frac{dv}{dy} = -2b, \quad \frac{du}{dy} = 0, \quad \frac{dv}{dx} = 0.$$

Hence, if $a < b$, F is a maximum; and, if $a > b$, it is neither a maximum nor a minimum.

The third system, as may be shewn in like manner, corresponds to a maximum value of F, when $a > b$, and to neither a maximum nor a minimum when $a < b$.

If $a = b$, the two partial differential coefficients of F have a common factor, and the second and third system may be rejected at starting.

5. Take $$F = xy^2\,(a - x - y)^3.$$

Then $$\frac{dF}{dx} = y^2\,(a - x - y)^2.(a - 4x - y),$$

$$\frac{dF}{dy} = xy\,(a - x - y)^2.(2a - 2x - 5y).$$

We may omit the factor $(a-x-y)^2$, as being essentially positive, and disregard the factor y as a common factor of the two partial differential coefficients of F.

We have accordingly two systems, as derived from the other factors,

$$\begin{pmatrix} x=0 \\ y=a \end{pmatrix}, \quad \begin{pmatrix} x=\frac{1}{6}a \\ y=\frac{1}{3}a \end{pmatrix}.$$

Taking the first system, we have

$$u=a-4x-y, \quad v=-3x;$$

whence $\dfrac{du}{dx}=-4, \quad \dfrac{dv}{dy}=0, \quad \dfrac{du}{dy}=-1, \quad \dfrac{dv}{dx}=-3.$

The criterion (5) therefore rejects this system. It may be observed that Lagrange's criterion would fail in this case.

Taking the second system, we have

$$u=2(a-4x-y), \quad v=2a-2x-5y,$$

$$\frac{du}{dx}=-8, \quad \frac{dv}{dy}=-5, \quad \frac{du}{dy}=-2, \quad \frac{dv}{dx}=-2.$$

The criterion (5) is satisfied and F is a maximum: although Lagrange's criterion would be applicable in this case, it involves more labour.

6. The student may take, as an exercise, the expression

$$F=x^2y^3(a-2x-3y)^4.$$

7. To ascertain whether the system $x=0$, $y=0$, renders

$$F=x^4+y^4-2x^2+4xy-2y^2$$

a maximum or minimum.

In this case Lagrange's criterion fails, and also the criterion (5): moreover, O'Brien's method is very inconvenient. The following method of treating this question is convenient, and the principle may be applied usefully in many such cases:

$$\frac{Df}{dr}=(x^3-x+y)\,l+(y^3-y+x)\,m,$$

$$\frac{D^2f}{dr^2}=(3lx^2-l+m)\,l+(3my^2-m+l)\,m,$$

$$=-(l-m)^2, \quad \text{when } x=0 \text{ and } y=0.$$

Hence f and therefore F is a maximum, unless $m=l$.

Put $m = l$. Then

$$\frac{Df}{dr} = l(x^3 + y^3),$$

$$\frac{D^2f}{dr^2} = 3l^2(x^2 + y^2),$$

$$\frac{D^3f}{dr^3} = 6l^3(x + y),$$

$$\frac{D^4f}{dr^4} = 12l^4;$$

that is, f and therefore F is a minimum.

Hence F is, absolutely, neither a maximum nor a minimum.

8. The method of the preceding example will easily shew that, when $x = 0$, $y = 0$, the expression

$$x^4 + y^4 - 4axy^2$$

is neither a maximum nor a minimum.

9. The principles of simplification which I have developed in this essay, are applicable to functions of more variables than two. I shall not however dwell longer upon this subject, because I think that what I have written above is sufficient to direct the student to methods of abbreviation of work in the more complicated problem of three or more variables. I will confine myself to observing that the analogous modification of the criteria* in the case of three independent variables may be effected by replacing respectively,

$$\frac{dF}{dx}, \quad \frac{dF}{dy}, \quad \frac{dF}{dz},$$

by

$$u, \quad v, \quad w;$$

$$\frac{d^2F}{dx^2}, \quad \frac{d^2F}{dy^2}, \quad \frac{d^2F}{dz^2},$$

by

$$\frac{du}{dx}, \quad \frac{dv}{dy}, \quad \frac{dw}{dz};$$

and

$$2\frac{d^2F}{dy\,dz}, \quad 2\frac{d^2F}{dz\,dx}, \quad 2\frac{d^2F}{dx\,dy},$$

by

$$\frac{dv}{dz} + \frac{dw}{dy}, \quad \frac{dw}{dx} + \frac{du}{dz}, \quad \frac{du}{dy} + \frac{dv}{dx}.$$

December, 1860.

* See Gregory's *Examples*, or Moigno's *Leçons de Calcul Differentiel et de Calcul Intégral*, Tome I. p. 136.

ON SOME GENERAL THEOREMS IN THE CALCULUS OF OPERATIONS AND THEIR APPLICATIONS.

By James W. Warren, A.B.

I commence with the following theorem: If D denote $\frac{d}{dx}$ on the product of two functions of x, neither of which exceed the degree $n-1$, and if d_1 and d_2 denote $\frac{d}{dx}$ on each taken alone, then representing by F the product of these functions,

$$D^{n+a} - n.\frac{n-a}{n}.\frac{(n+a).(n+a-1)}{n.n-1}D^{n+a-2}.d_1d_2$$

$$+\frac{n.n-3}{1.2}.\frac{(n-a).(n-a-1)}{n.n-1}.\frac{(n+a)\dots(n+a-3)}{n.n-1.n-2.n-3}D^{n+a-4}(d_1d_2)^2$$

$$-\frac{n.n-4.n-5}{1.2.3}.\frac{(n-a).(n-a-1).(n-a-2)}{n.n-1.n-2}$$

$$\times\frac{(n+a)\dots(n+a-5)}{n\dots(n-5)}D^{n+a-6}(d_1d_2)^3 + \&c.$$

on $F=0$; for expand each term by itself, neglecting all powers of d_1 and d_2 above the $(n-1)^{\text{th}}$, the general terms evidently are

$$D^{n+a} = \&c. + A_r.d_1^{n+a-r}.d_2^r + \&c.,$$

$$D^{n+a-2}(d_1d_2) = \&c. + \frac{r.(n+a-r)}{(n+a).(n+a-1)}.A_rd_1^{n+a-r}d_2^r + \&c.,$$

$$D^{n+a-4}(d_1d_2)^2 = \&c. + \frac{r.r-1\,(n+a-r).(n+a-r-1)}{(n+a).(n+a-1).(n+a-2).(n+a-3)}.A_rd_1^{n+a-r}d_2^r$$

$$D^{n+a-6}(d_1d_2)^3 = \&c.,$$

.... &c. ....

where $\quad A_r = \frac{(n+a).(n+a-1)\dots(n+a-r+1)}{1.2.3\dots r}.$

Now multiply

$$D^{n+a-2}(d_1d_2) \text{ by } -(n-a).\frac{(n+a).(n+a-1)}{n.n-1};$$

$$D^{n+a-4}(d_1d_2)^2 \text{ by } \frac{n.n-3}{1.2}\cdot\frac{(n-a).(n-a-1)}{n.n-1}$$
$$\times\frac{(n+a).(n+a-1).(n+a-2).(n+a-3)}{n.n-1.n-2.n-3}\ \&c.,$$

and add up; therefore coefficient of $d_1^{n+a-r}d_2^{r}$ is found to be

$$A_r\left\{1-\frac{r}{1}\cdot\frac{n-a}{n}\cdot\frac{n+a-r}{n-1}\right.$$
$$\left.+\frac{r.r-1}{1.2}\cdot\frac{(n-a).(n-a-1)}{n.n-1}\cdot\frac{(n+a-r).(n+a-r-1)}{(n-1).(n-2)}-\&c.\ \&c.\right\},$$

and now multiply this by

$$\{n.n-1\ldots(n-a+1)\}.\{(n-1)\ldots(n+a-r+1)\},$$

and bear in mind that r and $n+a-r$ can never be greater than $n-1$, nor less than unity; therefore r is always greater than a; also a is always less than $n-2$; therefore our multiplier can never equal zero: for $n+a-r$ equal to $n-1$; and therefore $a=r-1$. We of course only multiply by

$$n.(n-1)\ldots(n-a+1).$$

The general result evidently equals

$$A_r.n.(n-1)\ldots(n-a+1).(n-1)\ldots n+a-r+1)$$
$$-\frac{r}{1}.(n-1)\ldots(n-a).(n-2)\ldots(n+a-r),$$

or if in both cases we write our multiplier equal to $f(x)$ and observe that x is of the degree $a+r-a-1=r-1$, our expression by aid of the well-known theorem

$$\Delta^n U_x = U_{x+n}-\frac{n}{1}U_{x+n-1}+\frac{n.n-1}{1.2}U_{x+n-2}-\&c.$$

becomes $\qquad = A_r.\Delta^r.f(x-r);$

but $f(x-r)$ is of the degree $r-1$; therefore $\Delta^r.f(x-r)$ equals zero. Hence this theorem is proved.

I may remark that if for shortness we write $d_1.d_2=O$, then supposing $a=0$, we get the *unique* expansion of D^n in terms of D and O, equals

$$-n.D^{n-2}.O+\frac{n.n-3}{1.2}.D^{n-4}.O^2-\frac{n.n-4.n-5}{1.2.3}D^{n-6}.O^3\ \&c.\ \&c.$$

I shall now demonstrate a theorem similar to this last for the product of three functions of x, one of which does

not exceed the degree n, and the other two the degree $n-1$; write for distinction

$$D' = d_1 + d_2 + d_3;\ D = d_1 + d_2;\ O' = d_1.d_2.d_3;\ O = d_1.d_2.$$

Hence we have the following general terms:

$$D'^{2n} = \&c. + B_s.D^{2n-s}d_3^s + \&c.,$$

$$D'^{2n-3}.O' = \&c. + s.\frac{s.(2n-s).(2n-s-1)}{2n.(2n-1).(2n-2)}.B_s.D^{2n-s-2}.O.d_3^s + \&c.,$$

$$D'^{2n-6}.O'^2 = \&c.$$

$$+\frac{s.s-1.(2n-s).(2n-s-1).(2n-s-2).(2n-s-3)}{2n.(2n-1).(2n-2).(2n-3).(2n-4).(2n-5)}.B_s.D^{2n-s-4}O.^2d_3^s,$$

$$\&c.\ \&c.,$$

where $$B_s = \frac{2n.(2n-1)\ldots(2n-s+1)}{1.2\ldots s}.$$

Now putting in former theorem $n+a=2n-s$ we get

$$D^{2n-s} - n.\frac{s}{n}\ \frac{(2n-3).(2n-s-1)}{n.n-1}\ D^{2n-s-2}.O$$

$$+\frac{n.n-3}{1.2}.\frac{s.s-1}{n.n-1}.\frac{(2n-s)\ldots(2n-s-3)}{n.n-1\ldots n-3}\ D^{2n-s-4}.O^2 - \&c. = 0;$$

hence, multiplying the general terms above by proper factors to suit this, and adding up, we get

$$\Big\{D'^{2n} - \frac{1}{n-1}.\frac{2n.(2n-1)(2n-2)}{n}.D'^{2n-3}.O'$$

$$+\frac{1}{1.2.(n-1).(n-2)}$$

$$\times\frac{2n.(2n-1).(2n-2).(2n-3).(2n-4).(2n-5)}{n.n-1}D'^{2n-6}.O'^2 - \&c.\ \&c.\Big\}$$

upon $F=0$.

I shall call for shortness this operator Γ, and having proved that the result of Γ on the product of three functions only one of which exceeds the degree $n-1=0$. I shall now show that Γ operating on $U_n.V_n.W_n$ where

$$U_n = A_n.x^n + A_{n-1}x^{n-1} + \&c.,$$

$$V_n = B_n.x^n + B_{n-1}x^{n-1} + \&c.,$$

$$W_n = C_n.x^n + C_{n-1}x^{n-1} + \&c.,$$

equals

$$(2n.2n-1.2n-2\ldots 1).(B_n.C_n U_n + A_n.C_n V_n + A_n.B_n.W_n),$$

for we know that

$$f\left(\frac{d}{dx}\right).\psi(x) = f\left\{\frac{d}{d(x+1)}\right\}\psi(x).$$

Let $\qquad \Gamma_x.x^n.x^n.x = A.x;$

therefore $\qquad \Gamma_{x+1}.x^n.x^n.x = A.x;$

therefore $\quad \Gamma_{x+1}(x+1)^n.(x+1)^n.(x+1) = A.(x+1),$

and bearing in mind that

$$\Gamma x^n.x^{n-\alpha}.x^{n-\beta} = 0,$$

we have

$$\Gamma_{x+1}(x+1)^n.(x+1)^n.(x+1) = \Gamma_{x+1}x^n x^n.x + \Gamma_{x+1}x^n.x^n.1;$$

therefore $\qquad A(x+1) = Ax + B;$

therefore $\qquad A = B,$

and so in general we may prove that

$$\Gamma x^n.x^n.x^r = (1.2.3\ldots 2n).x^r$$

where r does not exceed $n-1$.

And it is evident that

$$\Gamma(x+1)^n.(x+1)^n.(x+1)^n$$

equals

$$\Gamma x^n.x^n.x^n + 3\Gamma x^n.x^n.\left(n.x^{n-1} + \frac{n.n-1}{1.2}x^{n-2} + \&c.\right).$$

Since $\qquad \Gamma x^n.x^{n-\alpha}.x^{n-\beta} = 0;$

therefore remembering results so far,

$$A.\{(x+1)^n - x^n\} = 3B.(nx^{n-1} + \&c.);$$

therefore $\qquad A = 3B,$

i.e. $\qquad \Gamma x^n.x^n.x^n = 3.(1.2.3\ldots 2n).$

From all this the theorem stated immediately follows: since put

$$U_n = A_n.x^n + U_{n-1},$$
$$V_n = B_n.x^n + V_{n-1},$$
$$W_n = C_n.x^n + W_{n-1};$$

therefore as $\qquad \Gamma x^n.x^{n-\alpha}.x^{n-\beta} = 0,$

we get $\Gamma . U_n V_n W_n = A_n . B_n C_n . \Gamma x^n . x^n . x^n$

$$+ \Gamma x^n . x^n . \{ B_n C_n . U_{n-1} + A_n C_n V_{n-1} + A_n . B_n . W_{n-1} \};$$

therefore result is theorem stated: this theorem may, of course, also be deduced by remembering that when we operate on two functions of the n^{th} degree U_n and V_n, D^{2n} does not vanish, but becomes equal to

$$\frac{2n.(2n-1).(2n-2)\ldots(n+1)}{1.2.3\ldots n} . d_1^n . d_2^n,$$

and this symbol on $U_n . V_n . W_n$ manifestly equals

$$\{2n.(2n-1).(2n-2)\ldots 1\} . A_n . B_n . W_n .$$

Similar theorems may be deduced for Γ on the product of any three functions, but they are not quite so elegant.

I shall now give a short example of the use of this theorem. Take the Hessian of a curve of the n^{th} degree, it consists of sums of the form $U_{n-2} . V_{n-2} . W_{n-2}$, and we know that

$$\left(x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz} \right)^{n-2}$$

on a quantic of the degree $3(n-h)$ equals

$$\left(x \frac{d}{dx'} + y \frac{d}{dy'} + z \frac{d}{dz'} \right)^{2n+2-3h}$$

on same quantic when x, y, z are changed into x', y', z' multiplied by

$$\frac{(3n-3h).(3n-3h-1)\ldots(2n-3h+3)}{(3n-3h).(3n-3h-1)\ldots(n-1)} .$$

Let now $x=0$, $y=0$, $d_1 . d_2 . d_3 (HU)'$ plainly equals the Hessian of the first polar of the origin $(d_1 . d_2 . d_3)^2 . (HU')$ the Hessian of the second polar, and so on; therefore

$$\left\{ \frac{d^{2n-4}}{dz'^{2n-4}} - \frac{1}{n-3} . \frac{(2n-4).(2n-5).(2n-6)}{n-2} \frac{d^{2n-7}}{dz'^{2n-7}} . O + \&c. \right\} . (HU'),$$

i.e., what we have called Γ on HU' may be replaced by

$$\Delta^{n-2}(HU) - \frac{1}{(n-2).(n-3)} \Delta^{n-2}(H_1 U)$$
$$+ \frac{1}{1.2.(n-2).(n-3)^2.(n-4)} \Delta^{n-2}(H_2 U) - \&c.,$$

and this equals a linear function of the terms in HU; various

theorems thus result from Γ; before enumerating a few it will conduce to clearness to have a distinct notation for the polars of curves, we will write then

$$\Pi_r = \left(x.\frac{d}{dx'} + y.\frac{d}{dy'} + z.\frac{d}{dz'}\right)^r.U.\frac{1}{n.(n-1)...(n-r+1)},$$

where x, y, z are the running coordinates of U; Π_r is then generally called the $(n-r)^{\text{th}}$ polar of U; with regard to point $x'y'z'$ it might perhaps be more proper to call it the r^{th} polar, since the name is more expressive the curve being of r^{th} degree.

In conclusion, I shall merely write down the following theorems as evident from Γ:

$$\Pi_{n-r}.(HU) - \frac{n^2}{1.(n-1).(n-2)}\frac{(2n-2).(2n-3).(2n-4)}{(3n-6).(3n-7).(3n-8)}\Pi_{n-r}(H\Pi_{n-1}U)$$

$$+ \frac{n^2.(n-1)^2}{1.2.(n-1).(n-2)^2.(n-3)}.\frac{(2n-2)...(2n-7)}{(3n-6)...(3n-11)}\Pi_{n-r}(H\Pi_{n-2}U) - \&c.$$

equals cypher where r may be equal to or greater than four, and by changing U into $\Pi_{n-1}U$ or $\Pi_{n-2}U$, and n into $n-1$ or $n-2$, &c., we get analogous theorems; therefore we have in general $r-3$ equations between (what are commonly called) the r^{th} polars of HU, $H\Pi_{n-1}U$, $H\Pi_{n-2}U$, &c., &c.; by proceeding thus we may arrive at some interesting results. As my paper has at present extended to a greater length than I intended, I shall however only mention a few in conclusion.

Let us call the linear function of the terms of the Hessian before mentioned ΛU, it is evident then by last theorem but one that

$$\Pi_1.HU + \alpha.\Pi_1.H\Pi_{n-1}U + \beta.\&c.$$

equals some constant into $\Pi_1\Lambda U$ (where α, β, &c. are functions of n); and we may replace U by $\Pi_{n-1}U$ or $\Pi_{n-2}U$, &c.

Now Mr. Salmon has proved (vide *Quarterly Journal*, No. 12, p. 321) that $\Pi_1 HU$, $\Pi_1 H\Pi_{n-1}U$, &c., are identical; therefore $\Pi_1\Lambda U$, $\Pi_1\Lambda\Pi_{n-1}U$, &c. are identical with one another, and $\Pi_1 HU$ &c.

I should think a closer examination of the properties of the operating symbols we have called D and O may give rise to some corresponding results in the higher geometry.

Thus Mr. Salmon's Equation of the Tangential may be written:

$$\left\{\frac{d^{2n-4}}{dz^{2n-4}} - \frac{n-1}{(n-2)^2}.\frac{(2n-2).(2n-3).(2n-4)}{1.}\frac{d^{2n-7}}{dz^{2n-1}}.O + \&c.\right\}.HU,$$

and the evaluation of this operator on $z^r.z^s.z^t$ in a concise form would lead to a verification of the tangential; this I have done, for $z^n.z^n.z^n$ in some cases, and indeed seemingly it will be sufficient to evaluate the operator on this, as an equation exists between two, and sometimes three, of the other expanded operators on general subject $z^r.z^s.z^t$. This possibly is the shortest proof of the tangential for particular values of n; the general case I have not yet worked out by this means, and since I believe Mr. Cayley has verified it in general, (although not having seen his paper, I am unaware whether his method agrees in any measure with the above). I suppose there is no use in going over the same ground.

In fine, the following expressions are *à priori* evident:

$$\Pi_r(\Lambda U) = a\Pi_r(\Lambda\Pi_s U),$$

since $$\Lambda\Pi_s U = \Lambda\frac{d^{n-s}}{dz^{n-s}}.U = \frac{d^{n-s}}{dz^{n-s}}\Lambda U.$$

38, Trinity College, Dublin.
May 30, 1861.

ON CERTAIN PROPERTIES OF PRIME NUMBERS.

By the Rev. J. Wolstenholme.

The properties I propose to prove in this article, for any prime number n, > 3, are (1) that the numerator of the fraction

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}$$

when reduced to its lowest terms is divisible by n^2, (2) the numerator of the fraction

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(n-1)^2}$$

is divisible by n, and (3) that the number of combinations of $2n-1$ things, taken $n-1$ together, diminished by 1, is divisible by n^3. I discovered the last to hold, for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally. The method I employed is somewhat laborious, and I should be glad if some of your readers would supply a more direct proof. I must first prove the following Lemma, the result of which is well known.

Lemma. The series

$$n-\frac{n(n-1)}{1.2}\cdot\frac{1}{2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{3}-\dots$$
$$+(-1)^{n-2}\,n.\frac{1}{n-1}+\frac{(-1)^{n-1}}{n}\ \{\equiv f(n)\}$$

is, when n is integral, equal to

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}.$$

For, we have

$$f(n)=n-\frac{n(n-1)}{1.2}\cdot\frac{1}{2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{3}-\dots$$
$$+(-1)^{n-2}.\frac{n}{n-1}+(-1)^{n-1}.\frac{1}{n},$$

$$f(n-1)=(n-1)-\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{2}$$
$$+\frac{(n-1)(n-2)(n-3)}{1.2.3}\cdot\frac{1}{3}-\dots+(-1)^{n-2}.\frac{1}{n-1}.$$

Hence

$$f(n)-f(n-1)=1-(n-1).\frac{1}{2}+\frac{(n-1).(n-2)}{1.2.3}-\dots$$
$$+(-1)^{n-2}+(-1)^{n-1}.\frac{1}{n}$$
$$=\frac{1}{n}\{1-(1-1)^n\}=\frac{1}{n};$$

similarly,

$$f(n-1)-f(n-2)=\frac{1}{n-1},\ \&\text{c.}\dots f(2)-f(1)=\frac{1}{2},\ \text{and}\ f(1)=1,$$

whence we obtain

$$f(n)=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}.$$

Now, let n be a prime number, and therefore odd; and we have

$$f(n-1)=(n-1)-\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{2}+\dots$$
$$+(-1)^{r-1}\frac{\lfloor n-1}{\lfloor r\,\lfloor n-r}\cdot\frac{n-r}{r}+\dots-\frac{1}{n-1},$$

and, inverting,

$$f(n-1) = -\frac{1}{n-1} + \ldots$$

$$+ (-1)^r \frac{|\underline{n-1}}{|\underline{r}\,|\underline{n-r}} \cdot \frac{r}{n-r} + \ldots + n - 1,$$

therefore, by addition,

$$2f(n-1) = \frac{n^2-2n}{n-1} - \ldots$$

$$+ (-1)^{r-1} \cdot \frac{|\underline{n-1}}{|\underline{r}\,|\underline{n-r}} \cdot \frac{n(n-2r)}{r(n-r)} + \ldots$$

$$= \frac{M(n^2)}{|\underline{n-1}} - 2n \left\{ \frac{1}{n-1} - \frac{n-1}{2(n-2)} + \ldots \right.$$

$$\left. + (-1)^{r-1} \cdot \frac{|\underline{n-1}}{|\underline{r}\,|\underline{n-r}} \cdot \frac{1}{(n-r)} + \ldots \right\},$$

$M(p)$ denoting a multiple of p.

Now

$$\frac{|\underline{n-1}}{|\underline{r}\,|\underline{n-r}} = \frac{(n-1)\ldots(n-r+1)}{1.2\ldots r}$$

$$= \frac{M(n) + (-1)^{r-1}\,|\underline{r-1}}{|\underline{r}} = \frac{M(n)}{|\underline{r}} + \frac{(-1)^{r-1}}{r};$$

whence, $2f(n-1)$

$$= \frac{M(n^2)}{|\underline{n-1}} - 2n \left\{ \frac{M(n)}{|\underline{n-1}} + \frac{1}{(n-1)} + \frac{1}{2.(n-2)} + \frac{1}{3.(n-3)} + \ldots \right\}$$

$$= \frac{M(n^2)}{|\underline{n-1}} - 2\left\{ \left(1 + \frac{1}{n-1}\right) + \left(\frac{1}{2} + \frac{1}{n-2}\right) + \left(\frac{1}{3} + \frac{1}{n-3}\right) + \ldots + \left(\frac{1}{n-1} + \frac{1}{1}\right) \right\}$$

$$= \frac{M(n^2)}{|\underline{n-1}} - 2\,\{2f(n-1)\}.$$

This gives, finally,

$$6f(n-1) = \frac{M(n^2)}{|\underline{n-1}},$$

whence, the first property stated, if n be any prime number > 3.

For the second property, we have

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(n-1)^2}$$

$$= \{f(n-1)\}^2 - \Sigma_{r=1}^{r=n-1} \frac{1}{r} \left\{ \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{r-1} + \frac{1}{r+1} + \ldots + \frac{1}{n-1} \right\}.$$

Now $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r-1}+\frac{1}{r+1}+\ldots+\frac{1}{n-1}$

$$=\left(\frac{1}{1}+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\ldots+\frac{1}{n-r}$$

$$=\frac{n}{n-1}+\frac{n}{2.(n-2)}+\frac{n}{3.(n-3)}+\ldots+\frac{n}{(r-1).(n-r+1)}+\frac{1}{n-r}$$

$$=\frac{M(n)}{\lfloor n-1}+\frac{1}{n-r}.$$

Hence

$$\Sigma_{r=1}^{r=n-1}\frac{1}{r}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r-1}+\frac{1}{r+1}+\ldots+\frac{1}{n-1}\right)$$

$$=\frac{M(n)}{\lfloor n-1}+\frac{1}{1.(n-1)}+\frac{1}{2.(n-2)}+\ldots+\frac{1}{(n-r).1}$$

$$=\frac{M(n)}{\lfloor n-1}+\frac{1}{n}\left\{\left(\frac{1}{1}+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\ldots+\left(\frac{1}{n-1}+\frac{1}{1}\right)\right\}$$

$$=\frac{M(n)}{\lfloor n-1}+\frac{2}{n}f(n-1)$$

$$=\frac{M(n)}{\lfloor n-1}+\frac{2}{n}\cdot\frac{M(n^2)}{\lfloor n-1}$$

$$=\frac{M(n)}{\lfloor n-1}.$$

Therefore

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{(n-1)^2}=\{f(n-1)\}^2-\frac{M(n)}{\lfloor n-1}=\frac{M(n)}{(\lfloor n-1)^2},$$

which proves the second property.

The number of combinations of $2n-1$ things, taken $n-1$ together, is

$$\frac{(2n-1)(2n-2)\ldots(n+1)}{1.2\ldots(n-1)},$$

or

$$\frac{\{n+(n-1)\}\{n+(n-2)\}\ldots(n+1)}{\lfloor n-1}=\frac{n^{n-1}+\ldots+p_2n^2+p_1n}{\lfloor n-1}+1,$$

p_1, p_2 being the sums of the products of the $(n-1)$ numbers $1, 2, 3, \ldots, (n-1)$, taken respectively $(n-2)$ and $(n-3)$ together. Hence

$$p_1 = \underline{|n-1}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n-1}\right) = M(n^2),$$

and

$$p_2 = \frac{1}{2}\,\underline{|n-1}\,\Sigma_{r=1}^{r=n-1}\,\frac{1}{r}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r-1}+\frac{1}{r+1}+\ldots+\frac{1}{n-1}\right)$$

$$= \frac{1}{2}\,\underline{|n-1}\,\frac{M(n)}{\underline{|n-1}} = \frac{1}{2}\,M(n) = M(n).$$

The number of combinations is therefore $\dfrac{M(n^2)}{\underline{|n-1}}+1$, or $M(n^2)+1$, since n cannot disappear on reduction, or the number of combinations, diminished by unity, is a multiple of n^2.

DECOMPOSITION OF RATIONAL FRACTIONS.

By JOSEPH HORNER, M.A., of Clare College.

To determine *continuously* the partial fractions which correspond to irresolvible quadratic factors of the denominator; *i.e.*, such as cannot be resolved into two real simple ones.

If a rational proper fraction $\dfrac{f(x)}{F(x)(x-a)^n}$ when transformed to the scale of y {or $(x-a)$} be $\dfrac{g(y)}{G(y)\,y^n}$, and $\dfrac{g(y)}{G(y)}$, expanded by division in an ascending series up to the term involving y^n, be

$$A + A_1y + A_2y^2 + \ldots A_{n-1}y^{n-1} + \frac{R(y)\,y^n}{G(y)},$$

then $$\frac{f(x)}{F(x)(x-a)} = \frac{A}{(x-a)^n} + \frac{A}{(x-a)^{n-1}} + \ldots\ \frac{A_{n-1}}{x-a} + \frac{R(y)}{G(y)},$$

where $A, A_1, \ldots A_{n-1}$ are the numerators of the partial fractions arising out of the factor $(x-a)^n$ in the denominator of the proposed fraction, and $\dfrac{R(y)}{G(y)}$ is the proper fraction remaining, which if necessary may be dealt with in like manner.

The foregoing method has been employed in one or two elementary works, and that it has not become more general notwithstanding its decided arithmetical superiority to any other, may be owing to the want of a corresponding method for irresolvible quadratic factors. This desideratum it is here proposed to supply.

Let the fraction be $\dfrac{f(x)}{F(x)\{(x-a)^2+b\}^n}$, where b is positive.

Put $y=x-a$, and let $\dfrac{f(x)}{F(x)}$ become by transformation $\dfrac{g(y)}{G(y)}$. Multiply numerator and denominator by $G(-y)$, which is obtained from $G(y)$ by changing the signs of the odd powers of y. The fraction thus becomes $\dfrac{g(y)\,G(-y)}{G(y)\,G(-y)}$; of which the denominator, since it remains the same when y changes sign, contains no odd power of y. This fraction may therefore be arranged in the form $\dfrac{\lambda(y^2)+y\mu(y)^2}{\nu(y^2)}$.

Assume $z=y^2+b$, and transform $\lambda(y^2)$, $\mu(y^2)$, $\nu(y^2)$ into the scale of z. The last fraction now takes the form $\dfrac{L(z)+yM(z)}{N(z)}$, and may be expanded by division, as far as the two terms involving z^{n-1}, in the ascending form

$$A+A_1z+\ldots A_{n-1}z^{n-1}+\frac{R(z)\,z^n}{N(z)}$$
$$+y\left\{B+B_1z+\ldots B_{n-1}z^{n-1}+\frac{S(z)\,z^n}{N(z)}\right\}.$$

Therefore $\dfrac{g(y)}{G(y)}=A+By+(A_1+B_1y)\,z$

$$+\ldots(A_{n-1}+B_{n-1}y)\,z^{n-1}+\frac{R(z)+yS(z)}{N(z)}\,z^n.$$

Now $R(z)+yS(z)$, when expressed in terms of y, must be divisible by $G(-y)$; otherwise, if we transformed the last equation into terms of y and multiplied up by $G(y)$, we should have the integral function $g(y)$ equal to the sum of an integral and a fractional function of y, which is a contradiction.

Let then $R(z)+yS(z)$ reduce to $Q(y)\,G(-y)$. This makes

$$\frac{g(y)}{G(y)(y^2+b)^n}=\frac{A+By}{(y^2+b)^n}+\frac{A_1+B_1y}{(y^2+b)^{n-1}}+\ldots\frac{A_{n-1}+B_{n-1}y}{y^2+b}+\frac{Q(y)}{G(y)},$$

or, restoring the value of x,

$$\frac{f(x)}{F(x)\{(x-a)^2+b\}^n}=\frac{(A-Ba)+Bx}{\{(x-a)^2+b\}^n}+\frac{(A_1-B_1a)+B_1x}{\{(x-a)^2+b\}^{n-1}}$$
$$+\ldots\frac{(A_{n-1}-B_{n-1}a)+B_{n-1}x}{(x-a)^2+b}+\frac{g(x)}{F(x)},$$

where $\frac{g(x)}{F(x)}$ is manifestly a proper fraction, and may, if requisite, be farther resolved.

Example. Resolve into partial fractions, the fraction

$$\frac{x^8+3x^7-3x^6-25x^5-73x^4-108x^3-95x^2-44x+61}{(x^3+4)\{(x+1)^2+2\}^3}.$$

Put $y=x+1$; then the process of transformation to the scale of y is as follows:

1	3	−3	−25	−73	−108	−95	44	61
	−1	−2	5	20	53	55	40	4
1	2	−5	−20	−53	− 55	−40	− 4	65
	−1	−1	6	14	39	16	24	
1	1	−6	−14	−39	− 16	−24	20	
	−1	0	6	8	31	−15		
1	0	−6	− 8	−31	15	−39		
0	−1	1	5	3	28			
1	−1	−5	− 3	−28	43			
	−1	2	3	0				
1	−2	−3	0	−28				
	−1	3	0					
1	−3	0	0					
	−1	4						
1	−4	4						
	−1							
1	−5							

That is, the numerator becomes

$$y^8-5y^7+4y^6-28y^4+43y^3-39y^2+20y+65.$$

By the same method x^3+4 becomes

$$y^3-3y^2+3y+3=G(y).$$

Multiply numerator and denominator by

$$G(-y),\text{ or }-y^3-3y^2-3y+3,$$

and we obtain

$$\lambda(y^2) = 2y^{10} + 6y^8 + 53y^6 - 116y^4 - 372y^2 + 95,$$
$$\mu(y^2) = -y^{10} + 8y^8 + y^6 - 6y^4 + 121y^2 - 135,$$
$$\nu(y^2) = -y^6 + 3y^4 - 27y^2 + 9.$$

Transforming these to the scale of z, where $z = y^2 + 2$, we find

$$L(z) = 2z^5 - 14z^4 + 85z^3 - 450z^2 + 696z + 83,$$
$$M(z) = -z^5 + 18z^4 - 103z^3 + 260z^2 - 179z - 249,$$
$$N(z) = -z^3 + 9z^2 - 51z + 83.$$

Reversing the order of these and performing the division,

$$\frac{L(z)}{N(z)} = 1 + 9z + 0z^2 + \frac{5 - 5z + 2z^2}{N(z)}\, z^3,$$
$$\frac{M(z)}{N(z)} = -3 - 4z + z^2 - \frac{19 - 5z}{N(z)}\, z^3.$$

Therefore the fractions are

$$-\frac{3y-1}{(y^2+2)^3} - \frac{4y-9}{(y^2+2)^2} + \frac{y}{y^2+2};$$

or, restoring the value of y,

$$-\frac{3x+2}{(x^2+2x+3)^3} - \frac{4x-5}{(x^2+2x+3)^2} + \frac{x+1}{x^2+2x+3}.$$

The remainder

$$-\frac{(19-5z)\,y - 5 + 5z - 2z^2}{N(z)},$$

when expressed in terms of y, and reduced by the factor $-y^3 - 3y^2 - 3y + 3$, is

$$-\frac{2y-1}{y^3 - 3y^2 + 3y + 3}, \quad \text{or} \quad -\frac{2x+1}{x^3+4}.$$

Any method of resolving such an example as we have chosen must necessarily be long; but the foregoing will be found much shorter, simpler, and more direct than any hitherto used.

Everton Vicarage, near St. Neot's.
October, 1860.

ON COAXIAL CIRCLES.

By JOHN CASEY, Scholar of Trinity College, Dublin, and Head-Master of the National District Model School, Kilkenny.

IT is necessary to premise the following Lemmas before proceeding to the immediate subject of this paper.

Lemma 1. *If a quadrilateral* ABCD *inscribed in a circle* X *of a coaxial system has two opposite sides* AB, CD, *touching a circle* Y *of the system, then the sides* AC, BD *touch another cercle* Z *of the system* (*fig.* 1), *and the points of contact* a, b, c, d, *are in a right line.*

This is evident from the similar triangles *Aab*, *Dcd*, and the similar triangles *Bac*, *Cbd*, and the equality of the ratios *Aa* : *Ab*, *Dd* : *Dc*, *Ba* : *Bc*, *Cd* : *Cb*. Q. E. D.

From this theorem, which is well known, many important inferences can be drawn.

COR. 1. *It follows in a similar way, that the sides* AD, BC *touch another circle* V *of the system, and that the points of contact* e, f, *are in the same right line with the points* a, b, c, d. *But if we suppose* X, Y, Z *fixed while the quadrilateral changes position, that* V *will be variable.*

COR. 2. *If we suppose another circle* X′ *of the same system to intersect* AD, BC *in the points* A′, D′, B′, C′, *the lines* A′B′, C′D′ *are tangents to a circle* Y′, *and* A′C′, B′D′ *to another circle* Z′ *of the system, and the line of contacts coincide with the line* abcd. *This is evident from the preceding Lemma and Cor.*

COR. 3. *When* X′ *touches the line* BC, Y′ *and* Z′ *coincide. We shall see further on, that the centres of* Y′ *and* Z′ *form a system of points in involution. When* Y′ *and* Z′ *coincide, we shall, by analogy, call it the* double circle *of the system.*

Lemma 2. *If a variable chord* AB (*the reader can easily construct the figure*) *of a circle* X *touch another circle* Y, *the velocity of the point* A : *velocity of* B :: *velocity acquired by a particle in falling from* A *to the radical axis of* X *and* Y : *velocity acquired in falling from* B *to the radical axis.*

Demonstration. Let a be the point of contact, then since a is the instantaneous centre of rotation, and the tangents at the points A, B, make equal angles with AB, velocity of A : velocity of B :: Aa : Ba :: the square root of the perpendicular from A to the radical axis : square root of the perpendicular from B to the radical axis :: velocity acquired in falling from A to the radical axis : velocity acquired in falling from B to the radical axis. Q. E. D.

COR. 1. *For the purpose of comparing the velocity of the point of contact* a *with the velocity of the point* B, let $A'B'$ be a position of AB infinitely near the position AB, and let a' be the point of contact of $A'B'$, and C the intersection of AB and $A'B'$. Now the arc aa' = angle C.radius of Y, but $C = \dfrac{AA' + BB'}{2 \text{ radius of } X}$; therefore $aa' = \dfrac{AA' + BB'}{2 \text{ radius of } X}$.radius of Y, but aa', AA', BB' are proportional to the velocities of the points a, A, B, respectively; hence velocity of a : velocity of A + velocity of B :: radius of Y : 2 radius of X, and velocity of A + velocity of B : velocity of B :: AB : aB; therefore velocity of a : velocity of B :: AB.radius of Y : aB.2 radius of X.

COR. 2. *From Cor.* 1 *it can be easily proved if* A″B″ *be another chord of* X *touching* Y *in* a″, *and if the envelope of* B″B *be a circle of the same coaxial system as* X *and* Y, *that the velocity of* a : *velocity of* a″ :: AB : A″B″.

Observation. Poncelet's Theorem (*see Propriétés Projectives,* p. 323) *follows as an immediate inference from either Lemma* 1 *or* 2, *and we shall find Cor.* 2 *to Lemma* 2 *of use in proving an important theorem. For, arriving at our next Lemma, it is necessary to give a special proof of a particular case of Poncelet's Theorem, from which proof the Lemma will follow as an inference.*

Particular Case of Poncelet's Theorem. If a variable triangle inscribed in a circle X *has two of its sides touching another circle* Y, *the envelope of the remaining side is a circle* (*Propriétés Projectives*, p. 323).

Demonstration. Describe a circle Z concentric with Y such that a triangle $AB'C'$ (fig. 2) described about it may be inscribed in X, let S be the common centre of Y and Z, join AS and produce it to meet X in E, let O be the centre of X, and draw its diameter ED. Now it is evident that BC is parallel to $B'C'$, and that EC : EC' :: radius of Y : radius

of Z, and $EC^2 : EC''^2 :: EF : EG :: OO' : OS$ [G being the point of contact of Z, and $B'C'$ and FO' being drawn parallel to EO]. Therefore the ratio of $OO' : OS$ is given; hence the point O' is given, and because OE and SG are each given, $O'F$ is given, and the circle described from O' as centre with $O'F$ as radius touches BC. Hence the envelope of BC is a circle. Q. E. D.

Lemma 3. *If from the point* a *where the chord of contact* bc *cuts* AS (*fig.* 2) aO″ *be drawn parallel to* OE, *and from the point of contact* F, FD *perpendicular to* bc, Fd *will pass through* H *a centre of similitude of the circles whose radii are* O′F *and* O″a, *and the points* S *and* H *are the foci of the conic which is the envelope of* bc.

Demonstration. From the similar triangles OSE, $O'HF$, we have

$$OE.O'H = O'F.OS = OE.O'S + OO'.SG;$$

therefore $$OE.SH = OO'.SG.$$

Again, $$AS.SE = 2OE.SG,$$

and $$AS.Sa = (\text{radius of } Y)^2;$$

therefore $2OE.SG : (\text{radius of } Y)^2 :: SE : Sa :: OS : SO''$;

therefore $OS : 2SO'' :: OE.SG : (\text{radius of } Y)^2$,

and $OO' : OS :: (\text{radius of } Y)^2 : SG^2$ (see last Dem.);

therefore by compounding,

$$OO' : 2SO'' :: OE : SG :: OO' : SH;$$

therefore $2SO'' = SH$, and $SO'' = O''H$.

Now from similar triangles

$$O'F : O''a :: O'H : SO'' :: O'H : O''H.$$

Therefore H is a centre of similitude, and it is also evident that S and H are the foci of the conic. Q. E. D.

COR. *If* X *and* Y *be two circles so related that a triangle* ABC *may be inscribed in* X *and circumscribed to* Y, *and if* a, b, c *be the points of contact on* Y, *the three perpendiculars of the triangle* abc *intersect in a given point, namely, one of the foci of the reciprocal of* X *with respect to* Y. This is evident from the foregoing.

Having established the foregoing Lemmas, we are prepared to proceed with the special subject of this paper.

1. *If there be three coaxial circles* X, Y, Z, *and from any point of* X *if tangents be drawn to* Y *and* Z *touching them in* B *and* C, *the envelope of* BC *is a conic section.*

Demonstration. Let S, O, O' (the reader can easily construct the fig.) be the centres of the circles, R, R' the radii of Y and Z, and OP, $O'P'$ perpendiculars from O, O' on BC. Now we have from a known property of coaxial circles,

$$SO:SO'::AB^2:AC^2::\cos^2 O'CP':\cos^2 OBP::\frac{R'^2-O'P'^2}{R'^2}:\frac{R^2-OP^2}{R^2},$$

hence $\quad SO.R'^2 : SO'.R^2 :: R'^2 - O'P'^2 : R^2 - OP^2$;

therefore

$$SO'.R^2.O'P'^2 - SO.R'^2.OP^2 = OO'.R^2.R'^2 = \text{a constant},$$

and, since $SO'.R^2$ and $SO.R'^2$ are given, let them be denoted by m and n, and the question is reduced to the following:— O′ *and* O *are two given points, it is required to find the envelope of a line such that* m *times the square of the perpendicular from* O′ *on it minus* n *times the square of the perpendicular from* O *may be given.*

Solution. Produce OO' (fig. 3) to L, and make

$$OL : O'L :: m : n,$$

and draw ON, OM parallel to PP', and take LQ a mean proportional between LN and LM, draw QK parallel and KI perpendicular to PP'. Now it is easy to prove that

$$mO'P'^2 - nOP^2 = mLM^2 - nLN^2 + (m-n)\,HL^2,$$

and that $\quad mLM^2 - nLN^2 = -(m-n)\,LQ^2$;

therefore $\quad mO'P'^2 - nOP^2 = (m-n)\,(HL^2 - LQ^2)$;

therefore $\quad HL^2 - LQ^2 = OO'.R^2.R'^2 \div m - n.$

Again, $\quad OL : O'L :: m : n$;

therefore $\quad OO' \div m - n = OL \div m,$

and we have

$$HL^2 - LQ^2 = OL.R^2.R'^2 \div m = OL.R'^2 \div SO';$$

therefore $\quad HL^2 - LQ^2 = \text{constant},$

$$KQ^2 + LQ^2 = KL^2 = \text{constant};$$

therefore $\quad HL^2 + KQ^2$ or $LI^2 = \text{constant},$

and the locus of I is a circle whose centre is L, and because K is a given point, and IH perpendicular to KI, the envelope of IH is a conic section whose foci are K, K' (LK' being made equal to LK). Q. E. D.

Observation. We have given the foregoing solution on account of its special use for the establishing the subsequent theorems, but the question may be done very concisely by dividing OO' internally and externally in the ratio of the square root of m to the square root of n, and it is easily proved that the product of the perpendiculars from the points of section is given; hence the envelope is a conic. Q. E. D.

COR. 1. *If the parts of a variable line intercepted by two given circles bear a constant ratio, the envelope of the line is a conic section.*

COR. 2. *If from the centres of two given circles perpendiculars be let fall on a variable line, and if tangents from the feet of the perpendiculars to the circles bear a constant ratio, the envelope of the line is a conic section.*

COR. 3. *If a variable line intersect one of two circles, and the intercept made on it by that circle bear a constant ratio to the tangent to the other from the foot of a perpendicular from its centre on the line, the envelope of the line is a conic.*

2. When PP' has a position parallel to OO', it is evident that LH will be the semiconjugate axis, and that its square is $OL.R'^2 \div SO'$, and since $KL^2 = OL.O'L$, we have

$$\text{semiconj}^2 : KL^2 :: R'^2 : SO'.O'L,$$

in like manner $\text{semiconj}^2 : KL^2 :: R^2 : SO.OL$;

therefore $R^2 : R'^2 :: SO.OL : SO'.O'L.$

Now if W, W' be the limiting points $R^2 = OW.OW'$, and $R'^2 = O'W.O'W'$; therefore

$$OW.OW' : O'W.O'W' :: SO.OL : SO'.O'L.$$

Hence the three pair of points S.L, OO', W.W' *are in involution.* Q. E. D.

COR. *The radical axis of this system intersects the line* SW' *in a point which is the centre of the circle of similitude of the circles whose centres are* O *and* O', *that is, of the*

circle described upon the distance between their centres of similitude, and its limiting points are the centres of similitude of O *and* O′.

This will easily appear by considering the point S to be at infinity.

3. Upon SL, OO', $W.W'$ as diameters describe circles; these will be coaxial, let their centres be C'', C', C. Now

$$OW.OW' : SO.OL : C'C : C'C'';$$

therefore $\quad$ semiconj$^2 : KL^2 :: C'C : C'C''$;

therefore

$$\text{semitrans}^2 : KL^2 :: C''C : C''C' :: LW.LW' : OL.O'L,$$

but $\quad OL.O'L = KL^2$;

therefore $\quad$ semitrans$^2 = LW.LW'$,

and because $W.W'$ are the limiting points of the original system $LW.LW'$ = half the square of the radius of the circle of that system whose centre is L; therefore semitrans = radius of the circle of the original system whose centre is L, *hence the circle described upon the transverse axis of the conic is a circle of the system* (see fig. 4). Q. E. D.

4. It has been (see fig. 4) proved that

$$\text{semitrans}^2 KL^2 :: C''C : C''C',$$

but $\quad C''C : C''C' :: SW.SW' : SO.SO'$;

therefore $SO.SO' \div SW.SW'$ = square of eccentricity of the conic, but $SW.SW'$ = square of radius of X. Hence we are enabled, given the centres of X, Y, Z, (see Art 1) to know the species of the envelope,

for if $SO.SO' <$ (radius of X)^2 it is an ellipse,

" $\quad SO.SO' =$ (radius of X)^2 " a parabola,

" $\quad SO.SO >$ (radius of X)^2 " a hyperbola.

Definitions. For the sake of avoiding tedious periphrases I have found it necessary to use the following nomenclature:

(1) If from a point in the circumference X of a coaxial system tangents be drawn to two other circles Y and Z of the system touching them in B and C, we have seen in Art. 1 that the envelope of BC is a conic section. This

conic I shall call the *reciprocant* of X with respect to Y and Z. I call it *reciprocant* in order to distinguish it from the ordinary reciprocal of one circle with respect to another, to which, indeed, it is nearly allied, the latter being a species of the former.

(2) *Triangular system of circles.* Three circles of a system which are the envelopes of the three sides of a variable triangle inscribed in another circle of the system, I shall call a triangular system of circles (see *Propriétés Projectives*).

(3) Two circles (Y and Z, see fig. 5) of a triangular system with respect to X being given, the third circle may be either of two circles as appears from the diagram. The circle, such as W, I shall call the *complementary* circle, and W' the *sub-complementary*, and W and W' *duplicates of each other.*

5. *If three circles* Y, Z, W *form a triangular system with respect to* X, *the centres of* Y, Z, W *and the centres of the reciprocants of* X *with respect to each two of them form a system of points in involution.*

Demonstration. Let O, O', O'' be the centres of Y, Z, W, R, R', R'' their radii, L, L', L'' the centres of the reciprocants. Now from the demonstration of Art. 2, we have

$$SO.OL : SO'.O'L :: R^2 : R'^2,$$

$$SO'.O'L' : SO''.O''L' :: R'^2 : R''^2,$$

$$SO''.O''L'' : SO.OL'' :: R''^2 : R^2.$$

Hence $$OL.O'L'.O''L'' = O'L.O''L'.OL'',$$

and the proposition is proved.

6. *The foci of the reciprocants in the last proposition form a system of points in involution.*

Demonstration. Let the reciprocants be denoted by Σ, Σ', Σ'', and the foci by K, K'; K_1, K_1'; K_2, K_2'; and let the circle described on KK' intersect the circle described on K_1K_1' in the points P, P'. Now

$$SO.OL : SO'.O'L :: R^2 : R'^2;$$

therefore $$OL : O'L :: R^2 \div SO : R'^2 \div SO',$$

but $$OL : O'L :: OK^2 : O'K^2 :: OP^2 : O'P^2;$$

therefore $$OP^2 : O'P^2 :: R^2 \div SO : R'^2 \div SO';$$

in like manner

$$O'P^2 : O''P^2 :: R'^2 \div SO' : R''^2 \div SO'';$$

therefore

$$OP^2 : O''P^2 :: R^2 \div SO : R''^2 \div SO'' :: OK_2 : O''K_2^2;$$

hence the circle described on K_2K_2' passes through P, in like manner it passes through P'; therefore these circles are coaxial, and the proposition is proved.

7. *The three centres* O, O′, O″ *of the circles* Y, Z, W, *and the foci* K, K_1, K_2' *of the reciprocants* Σ, Σ′, Σ″ *form a system in involution.*

Demonstration. It has been proved in the last article that

$$OK^2 : O'K^2 :: R^2 \div SO : R'^2 \div SO'.$$

Now, attending to the convention relative to the signs of lines, namely, that lines measured in the same direction from a common origin have *like signs*, and in a contrary direction *unlike signs;* we have, by extracting the square root,

$$OK : -O'K :: R + \surd(SO) : R' \div \surd(SO'),$$

in like manner

$$O'K_1 \div -O''K_1 :: R' \div \surd(SO') : R'' \div \surd(SO'').$$

Again, taking the foci K_2 and K_2' of the reciprocant Σ″, we shall find that $O''K_2'$ and OK_2' are measured in the same direction; hence we have

$$O''K_2' : OK_2' :: R'' \div \surd(SO'') : R \div \surd(SO);$$

therefore $\quad OK.O'K_1.O''K_2' = O'K.O''K_1.OK_2',$

and therefore the points are in involution. Q. E. D.

Cor. 1. In like manner the points

$$\left.\begin{array}{ll} & O,\ O',\ O'',\ K_1,\ K_2,\ K' \\ \text{and} & O,\ O',\ O'',\ K_2,\ K,\ K_1' \end{array}\right\} \text{are in involution.}$$

Cor. 2. The points O, O', O'', K', K_1', K_2' are in involution. For we have proved that

$$O''K_2' : OK_2' :: R'' \div \surd(SO'') : R \div \surd(SO),$$

and by combining this with the two other similar proportions the proposition is evident.

8. We have seen in Cor. 2, Lemma 1, that if a quadrilateral $ABCD$ inscribed in a circle X of a coaxial system (fig. 1) whose sides AB, CD touch a circle Y, and AC, BD touch another circle Z of the system, has its sides AD and BC intersected in the points A', D', B', C' by a circle X' of the same system, that the sides $A'B'$, $C'D'$ touch a circle Y', and $A'C'$, $B'D'$ another circle Z' of the system, and the lines of contact are coincident; also, that when X' coincides with the envelope of BC, that Y' and Z' coincide, and we have called that circle of the system the double circle. From this it is evident that the reciprocant of X with respect to Y and Z is also the reciprocant of X' with respect to Y' and Z'; and again, the reciprocal of the circle which is the envelope of BC with respect to the double circle; that is, *the reciprocant of* X *with respect to* Y *and* Z *is the reciprocal of the complementary circle with respect to the double circle.* Q. E. D.

COR. *The centres of the circles* Y', Z' *form a system of points in involution. This is evident since they are harmonic conjugates with respect to the foci of the reciprocant.*

9. From Arts. 2 and 8 we derive the following construction for finding the centre of the complementary circle of a triangular system when the centres O, O' of the two other circles and the centre S of the circumscribing circle are given.

Let W, W' (the reader can form the figure) be the limiting points of the system. Find L a sixth point in involution with W, W'; O, O'; and S; then L is the centre of the reciprocant; take LK a mean proportional between OL and $O'L$, then K is a focus of the reciprocant. Again, find O'' such that a circle described on $O''L$, and another described on W, W', may have K as one of their limiting points; O'' is the point required.

Demonstration. It was proved, Art. 8, that the reciprocant is the reciprocal of the complementary circle with respect to the double circle; hence, in order to prove the foregoing construction, all that is required to be shewn is, that if we reciprocate one circle with respect to another, the centre of the reciprocating circle is one of the double points of a system in involution determined by the limiting points of the two circles, the centre of the reciprocated circle and the centre of the resulting conic. Now this is seen at once from Art. 2, by supposing the points O, O' to coincide. Q. E. D.

Cor. 1. K *is a centre of similitude of the circles of the system whose centres are* L *and* O″, *that is, of the circle described on the transverse of the reciprocant, and the complementary circle of the triangular system. This is evident from looking at fig.* 2. In that figure S is a centre of similitude of the circle X, and the circle whose radius is $O''a$, and the proposition here stated is the same.

Cor. 2. *The other focus* K′ *of the reciprocant is a centre of similitude of the sub-complementary circle of the triangular system, and the circle on the transverse axis of the reciprocant.* This follows at once from Lemma 3.

Cor. 3. *Hence the foregoing construction for finding the centre of the complementary circle will determine the centre of the sub-complementary circle if in it we use* K′ *instead of* K.

Cor. 4. *Hence if the sub-complementary circle be one of the limiting points,* K′ *will be the same point, and the corresponding directrix of the reciprocant will pass through the other limiting point.*

10. *If* X *and* Y (*fig.* 1) *be two circles, and* BC *a chord of* X *whose envelope is a circle of the same system with* X *and* Z, *and* Bc, Cb *tangents to* Z *from* B *and* C, *the envelope of the chord of contact* bc *is a conic section.*

Demonstration. Produce Bc, Cb, to D and A. Join AB, CD. Then the proposition is evident from Poncelet's Theorem, Lemma 1 and Art. 1. Q. E. D.

11. If $ABCD$ be a quadrilateral inscribed in a circle X of a coaxial system whose sides AB, CD (fig. 6) touch a given circle Y of the system, and consequently, whose sides AC, BD touch another circle Z of the system, then if Z be given, the envelope of BC is given. From the point of contact t of BC with its envelope let fall a perpendicular tm on the line of contact $abcd$. Then since the envelope of bc is the reciprocal of the envelope of BC with respect to the double circle of the system Y, Z, from the known property of reciprocation tm produced passes through the centre of the double circle, that is, through a focus of the conic, *and the locus of the point* m *is the circle described on its transverse, which has been proved to be a circle of the system.* Again, since the envelope of AD is the sub-

complementary of Y and Z with respect to X, the perpendicular $t'm'$ let fall from t', the point of contact of AD with its envelope on bc passes through the other focus of the conic, Lemma 3, *and the locus of* m′ *is the circle on the transverse.*

12. From the last Article we have the following theorems:— *If* ABC *be a triangle inscribed in a circle* X, *whose sides are tangents to a triangular system of circles* Y, Z, W, *and if the points of contact on the sides of* ABC *be joined forming a new triangle* abc, *the loci of the feet of the perpendiculars of* abc *are circles of the system, and their centres and centres of* Y, Z, W, *form a system of points in involution* (Art. 4).

And again, each of the perpendiculars passes through a given point, and the three given points through which they pass, and the centres of Y, Z, W, *form a system of points in involution* (Art. 7).

13. The circle described on the transverse axis of a conic having double contact with it, the anharmonic ratio of the four points m (fig. 6) is equal to the anharmonic ratio of the four corresponding points on the conic, and this again equal to the anharmonic ratio of the corresponding four points t on BC, BC being supposed to take four different positions along its envelope. Again, the anharmonic ratio of the corresponding four points t' on AD is equal to the anharmonic ratio of the four points m'. Since $t'm'$ passes through a centre of similitude of the circle on the transverse and the envelope of AD, *hence the anharmonic ratio of the four points* t *on* BC *is equal to the anharmonic ratio of the four corresponding points* t′ *on* AD. Q. E. D.

(*To be continued.*)

ON CERTAIN SYSTEMS OF CURVES OF THE THIRD DEGREE PASSING THROUGH THE VERTICES AND THE INTERSECTIONS OF OPPOSITE SIDES AND DIAGONALS OF A GIVEN QUADRILATERAL.

By Samuel Roberts, M.A.

1. Let the intersections be the vertices of the triangle of reference. The coordinates of the vertices of the quadrilateral may be represented by

$$\begin{array}{ccc} m, & n, & p, \\ -m, & n, & p, \\ n, & -m, & p, \\ m, & n, & -p, \end{array}$$

and the form of the equation for the systems of cubics passing through these points, and $(\alpha\beta)$, $(\alpha\gamma$, $(\beta\gamma)$ is

$$k_1\left\{\left(\frac{\beta}{n}\right)^2-\left(\frac{\gamma}{p}\right)^2\right\}\frac{\alpha}{m}$$
$$+k_2\left\{\left(\frac{\gamma}{p}\right)^2-\left(\frac{\alpha}{m}\right)^2\right\}\frac{\beta}{n}+k_3\left\{\left(\frac{\alpha}{m}\right)^2-\left(\frac{\beta}{n}\right)^2\right\}\frac{\gamma}{p}=0 \;\ldots\ldots(a).$$

In accordance with a theorem of Maclaurin's, the polar conic of $\left(\frac{m}{k_1}, \frac{n}{k_2}, \frac{p}{k_3}\right)$ (a point on the curve determined by the particular values of k_1, k_2, k_3) is

$$\left(\frac{k_3}{k_1}-\frac{k_1}{k_3}\right)\frac{\alpha}{m}\cdot\frac{\gamma}{p}+\left(\frac{k_2}{k_3}-\frac{k_3}{k_2}\right)\frac{\gamma}{p}\cdot\frac{\beta}{n}+\left(\frac{k_1}{k_2}-\frac{k_2}{k_1}\right)\frac{\beta}{n}\cdot\frac{\alpha}{m}=0,$$

and the polar conic of (mk_1, nk_2, pk_3) (also a point on the curve) is

$$k_1^2\left\{\left(\frac{\beta}{n}\right)^2-\left(\frac{\gamma}{p}\right)^2\right\}+k_2^2\left\{\left(\frac{\gamma}{p}\right)^2-\left(\frac{\alpha}{m}\right)^2\right\}+k_3^2\left\{\left(\frac{\alpha}{m}\right)^2-\left(\frac{\beta}{n}\right)^2\right\}=0.$$

Consequently the four tangents at the vertices of the quadrilateral meet in a point (mk_1, nk_2, pk_3) on the curve,

and the tangents at $(\alpha\beta)$, $(\beta\gamma)$, $(\gamma\alpha)$ and (mk_1, nk_2, pk_3) meet at a point $\left(\frac{m}{k_1}, \frac{n}{k_2}, \frac{p}{k_3}\right)$ on the curve.

2. If $\alpha_1, \beta_1, \gamma_1$ satisfy (a), it is also satisfied by $\frac{m^2}{\alpha_1}, \frac{n^2}{\beta_1}, \frac{p^2}{\gamma_1}$; *i.e.*, if a system of curves (a) pass through α, β, γ, they pass also through the fixed point $\frac{m^2}{\alpha_1}, \frac{n^2}{\beta_1}, \frac{p^2}{\gamma_1}$. This is evident, by substitution.

If, therefore, the locus of $(\alpha_1\beta_1\gamma_1)$ is taken as $F(\alpha_1\beta_1\gamma_1)=0$, that of the other or ninth common point will be

$$F\left(\frac{m^2}{\alpha_1}, \frac{n^2}{\beta_1}, \frac{p^2}{\gamma_1}\right)=0.$$

Hence if the locus of $\alpha_1\beta_1\gamma_1$ be a right line, that of the corresponding point will be a conic through the vertices of the triangle of reference.

Since the intersections of tangents at the four vertices of the quadrilateral, and of tangents at $(\alpha\beta)$, $(\beta\gamma)$, $(\alpha\gamma)$ are reciprocal, the relation above indicated exists between their respective loci.

The equation (a) may be put into the form

$$\left(\frac{\alpha}{m}-\frac{q_2}{k}\cdot\frac{\beta}{n}\right)\left(\frac{\beta}{n}-\frac{q_1}{k}\cdot\frac{\gamma}{p}\right)\left(\frac{\gamma}{p}-\frac{q_2}{k}\cdot\frac{\alpha}{m}\right)$$
$$=\left(\frac{\beta}{n}-\frac{q_2}{k}\cdot\frac{\alpha}{m}\right)\left(\frac{\gamma}{p}-\frac{q_1}{k}\cdot\frac{\beta}{n}\right)\left(\frac{\alpha}{m}-\frac{q_2}{k}\cdot\frac{\gamma}{p}\right)\ldots\ldots\ldots(b).$$

If k remain a variable parameter, (b) represents a system of curves through the seven points of the systems (a), and through (mq_1, nq_2, pq_3). The curve (b) also passes through

$$mq_2,\quad nk,\quad pq_1,\quad \frac{m}{q_2},\quad \frac{n}{k},\quad \frac{p}{q_1},$$
$$mk,\quad nq_3,\quad pq_2,\quad \frac{m}{k},\quad \frac{n}{q_3},\quad \frac{p}{q_2},$$
$$mq_2,\quad nq_1,\quad pk,\quad \frac{m}{q_2},\quad \frac{n}{q_1},\quad \frac{p}{k},$$

and as k varies, the lines joining these points two and two, envelope determinate conics; for their equations contain k in the second degree.

(*b*) may be arrived at as follows:—The lines

$$\gamma_0\left(\frac{\alpha}{m}-\frac{q_3}{k}\frac{\beta}{n}\right)=\alpha_0\left(\frac{\gamma}{p}-\frac{q_1}{k}\frac{\beta}{n}\right),$$

$$\beta_0\left(\frac{\gamma}{p}-\frac{q_2}{k}\frac{\alpha}{m}\right)=\gamma_0\left(\frac{\beta}{n}-\frac{q_3}{k}\frac{\alpha}{m}\right),$$

$$\beta_0\left(\frac{\beta}{n}-\frac{q_1}{k}\frac{\gamma}{p}\right)=\beta_0\left(\frac{\alpha}{m}-\frac{q_2}{k}\frac{\gamma}{p}\right),$$

pass respectively through

$$\left.\begin{matrix}(1)\ m\alpha_0, & 0, & p\gamma_0\\ (4)\ mq_3, & nk, & pq_1\end{matrix}\right\},\quad \left.\begin{matrix}(2)\ 0, & n\beta_0, & p\gamma_0\\ (5)\ mk, & nq_3, & pq_2\end{matrix}\right\},\quad \left.\begin{matrix}(3)\ m\alpha_0, & n\beta_0, & 0\\ (6)\ mq_2, & nq_1, & pk\end{matrix}\right\},$$

and right lines from $(\alpha\gamma)$, $(\beta\gamma)$, $(\beta\alpha)$ through (1), (2), (3) respectively meet in $m\alpha_0$, $n\beta_0$, $p\gamma_0$.

Eliminating α_0, β_0, γ_0, we have (*b*), which represents therefore a locus such that if right lines be drawn from a point thereon through the fixed points (4), (5), (6), they will meet the sides of the triangle so that right lines from the vertices to the intersections will meet in one point.

3. If $m=n=p$, (*a*) represents the system of cubics passing through the vertices of a triangle and the centres of inscribed and escribed circles. In this case $\left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$ is the corresponding point to (α, β, γ).

There are three well known pairs of points, which have reciprocal coordinates.

The centre of the circumscribing circle of the triangle of reference, and the intersection of perpendiculars from the vertices on the sides are denoted by

$$\left.\begin{matrix}\cos A, & \cos B, & \cos C\\ \frac{1}{\cos A}, & \frac{1}{\cos B}, & \frac{1}{\cos C}\end{matrix}\right\}\ldots\ldots\ldots\ldots\ldots(1).$$

The intersection of bisectors of the sides from the vertices and the like intersection of the circumscribing triangle are denoted by

$$\left.\begin{matrix}\frac{1}{\sin A}, & \frac{1}{\sin B}, & \frac{1}{\sin C}\\ \sin A, & \sin B, & \sin C\end{matrix}\right\}\ldots\ldots\ldots\ldots\ldots(2),$$

and the coordinates of the circular points at infinity are of the forms

$$\left.\begin{array}{ccc} \alpha', & \beta', & \gamma' \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Consequently a cubic (a) $(m=n=p)$ may be found which shall pass through any two of these pairs of points.

4. The curve

$$\frac{\alpha(\beta^2-\gamma^2)}{\sin A}+\frac{\beta(\gamma^2-\alpha^2)}{\sin B}+\frac{\gamma(\alpha^2-\beta^2)}{\sin C}=0 \ldots\ldots. (c),$$

is a curve (a) $(m=n=p)$, and passes through the first two pairs of points just referred to. The tangents at the vertices of the triangle and $\left(\frac{1}{\sin A}, \frac{1}{\sin B}, \frac{1}{\sin C}\right)$ meet at $(\sin A, \sin B, \sin C)$; and the tangents from the point $\left(\frac{1}{\sin A}, \frac{1}{\sin B}, \frac{1}{\sin C}\right)$ touch at the centres of inscribed and escribed circles.

The tangent at $(\sin A, \sin B, \sin C)$ is

$$\frac{\sin^2 B-\sin^2 C}{\sin A}\alpha+\frac{\sin^2 C-\sin^2 B}{\sin B}\beta+\frac{\sin^2 A-\sin^2 B}{\sin C}=0,$$

which is the polar of the centre of the conic through the four centres and the point of contact, with regard to the triangle.

The curve (c) passes through the middle points of the sides of the triangle and the tangents at those points and at $(\sin A, \sin B, \sin C)$ meet in a point on the curve.

The form of (c) corresponding to (b) is

$$(\alpha-\cos C\beta)(\beta-\cos A\gamma)(\gamma-\cos B\alpha)$$
$$=(\beta-\cos C\alpha)(\gamma-\cos A\beta)(\alpha-\cos B\gamma)\ldots\ldots\ldots\ldots\ldots(c').$$

5. The form (c') is allied to the form

$$(\alpha+\cos C\beta)(\beta+\cos A\gamma)(\gamma+\cos B\alpha)$$
$$=(\beta+\cos C\alpha)(\gamma+\cos A\beta)(\alpha+\cos B\gamma)\ldots\ldots\ldots\ldots\ldots(d),$$

which represents a locus such that right lines from a point thereon perpendicular to the sides of the triangle cut those sides in points, right lines drawn from which to the vertices meet in one point.

The three fixed points being on the line at infinity, it follows that the three corresponding points and $(\alpha\beta)$, $(\alpha\gamma)$, $(\beta\gamma)$ lie on the same circle, as is plain in the general case by Pascal's theorem. The position of these points is determined by right lines from the vertices of the triangle of reference through the centre of the circumscribing circle.

6. Considered as a locus of the kind (b), curves of the third degree (a) $(m=n=p)$ take the form

$$(\alpha-k_3\beta)(\beta-k_1\gamma)(\gamma-k_2\alpha)=(\beta-k_3\alpha)(\gamma-k_1\beta)(\alpha-k_2\gamma),$$

and the factors $(\alpha-k_3\beta)$ and $(\beta-k_3\alpha)$ equated to cypher represent right lines equally inclined to the bisector $\alpha-\beta=0$; and the same applies to the other factors taken in corresponding pairs.

March, 1861.

THEORY OF GENERIC EQUATIONS.

By John Blissard, M.A.

(Continued from Vol. IV. p. 305.)

25. Since $\varepsilon^{(U+m)\theta}=\varepsilon^{-U\theta}$; therefore equating coefficients of θ^n, $(U+m)^n=(-U)^n$; therefore

$$f\{x+(U+m)\,\theta\}=f(x-U\theta),$$

let
$$x=-\tfrac{1}{2}m\theta;$$

therefore
$$f(U+\tfrac{1}{2}m)\,\theta=f\{-(U+\tfrac{1}{2}m)\,\theta\};$$

therefore
$$(U+\tfrac{1}{2}m)^{2n+1}=0.$$

Similarly
$$(V+\tfrac{1}{2}m)^{2n+1}=0.$$

When $m=1$, U becomes A, and V becomes B; therefore $(A+\frac{1}{2})^{2n+1}=0$, $(B+\frac{1}{2})^{2n+1}=0$, results which have been already obtained (see Art. 18).

26. Required to expand $\dfrac{\cos m\theta}{(\cos\theta)^m}$, $\dfrac{\sin m\theta}{(\sin\theta)^m}$, $\left(\dfrac{\theta}{\sin\theta}\right)^m\cos m\theta$, and $\left(\dfrac{\theta}{\sin\theta}\right)^m\sin m\theta$.

(1). From $\varepsilon^{U\theta} = \dfrac{2^m}{(\varepsilon^\theta + 1)^m}$,

$$\varepsilon^{U\theta\sqrt{(-1)}} + \varepsilon^{-U\theta\sqrt{(-1)}} = \frac{2^m}{\{\varepsilon^{\theta\sqrt{(-1)}} + 1\}^m} + \frac{2^m}{\{\varepsilon^{-\theta\sqrt{(-1)}} + 1\}^m}$$

$$= 2^m \left\{\frac{\varepsilon^{\frac{1}{2}m\theta\sqrt{(-1)}} + \varepsilon^{-\frac{1}{2}m\theta\sqrt{(-1)}}}{\{\varepsilon^{\frac{1}{2}\theta\sqrt{(-1)}} + \varepsilon^{-\frac{1}{2}\theta\sqrt{(-1)}}\}^m}\right\};$$

therefore $\quad 2\cos U\theta = 2^m \left\{\dfrac{2\cos\frac{1}{2}m\theta}{(2\cos\frac{1}{2}\theta)^m}\right\} = \dfrac{2\cos\frac{1}{2}m\theta}{(\cos\frac{1}{2}\theta)^m}$;

therefore putting 2θ for θ, $\cos 2U\theta = \dfrac{\cos m\theta}{(\cos\theta)^m}$.

Hence

$$\frac{\cos m\theta}{(\cos\theta)^m} = 1 - \frac{U_2(2\theta)^2}{1.2} + \frac{U_4(2\theta)^4}{1.2.3.4} - \frac{U_6(2\theta)^6}{1.2.3.4.5.6} + \&c. \ldots\text{(XIII.)}$$

(2). From

$$\varepsilon^{U\theta\sqrt{(-1)}} - \varepsilon^{-U\theta\sqrt{(-1)}} = \frac{2^m}{\{\varepsilon^{\theta\sqrt{(-1)}} + 1\}^m} - \frac{2^m}{\{\varepsilon^{-\theta\sqrt{(-1)}} + 1\}^m},$$

we similarly obtain

$$\frac{\sin m\theta}{(\cos\theta)^m} = -\sin 2U\theta$$

$$= -\left\{U_1(2\theta) - \frac{U_3(2\theta)^3}{1.2.3} + \frac{U_5(2\theta)^5}{1.2.3.4.5} - \&c.\right\}\ldots\ldots\text{(XIV.)}$$

(3). In like manner we obtain from $\varepsilon^{V\theta} = \dfrac{\theta^m}{(\varepsilon^\theta - 1)^m}$,

$$\left(\frac{\theta}{\sin\theta}\right)^m \cos m\theta = \cos 2V\theta$$

$$= 1 - \frac{V_2(2\theta)^2}{1.2} + \frac{V_4(2\theta)^4}{1.2.3.4} - \&c. \ldots\ldots\ldots\text{(XV.)}$$

(4). and $\left(\dfrac{\theta}{\sin\theta}\right)^m \sin m\theta = -\sin 2V\theta$

$$= -\left\{V_1(2\theta) - \frac{V_3(2\theta)^3}{1.2.3} + \&c.\right\}\ldots\ldots\ldots\text{(XVI.)}$$

Ex. In (XIII.) let $m = 2$; therefore

$$\frac{\cos 2\theta}{(\cos\theta)^2}\ (= 1 - \tan^2\theta) = 1 - \frac{U_2(2\theta^2)}{1.2} + \frac{U_4(2\theta)^4}{1.2.3.4} - \&c.,$$

where $U_n\ (m = 2)$ by (XI.),

$$= -2\left(\frac{2^{n+2} - 2}{n+2} B_{n+2} + \frac{2^{n+1} - 2}{n+1} B_{n+1}\right) = -2\,\frac{(2^{n+2} - 2)}{n+2} B_{n+2}\ (n \text{ even});$$

therefore $\tan^2\theta = -2\left\{\frac{2^4-1}{2}B_4\frac{(2\theta)^2}{1.2} - \frac{2^6-1}{3}B_6\frac{(2\theta)^4}{1.2.3.4}\right.$

$$\left.+\frac{2^8-1}{4}B_8\frac{(2\theta)^6}{1.2\ldots6} - \&c.\right\}.$$

27. Required to expand $\frac{\cos(n-m)\theta}{(\cos\theta)^{m+n}}$ and $\frac{\sin(n-m)\theta}{(\cos\theta)^{m+n}}$.

In $\varepsilon^{U\theta} = \frac{2^m}{(\varepsilon^\theta+1)^m}$ put x for ε^θ; therefore $x^U = \frac{2^m}{(x+1)^m}$. Differentiate n times, then

$$U.(U-1).(U-2)\ldots(U-n+1)\,x^{U-n}$$
$$= (-1)^n\,2^m.m.(m+1)\ldots(m+n-1).\frac{1}{(x+1)^{m+n}};$$

therefore $U.(U-1).(U-2)\ldots(U-n+1).x^U$

$$= (-1)^n.2^m.m.(m+1)\ldots(m+n-1).\frac{x^n}{(x+1)^{m+n}}.$$

For x put successively $\varepsilon^{2\theta\sqrt{(-1)}}$, $\varepsilon^{-2\theta\sqrt{(-1)}}$, then by taking the sum we have

$$U.(U-1).(U-2)\ldots(U-n+1).\cos 2U\theta$$
$$= (-1)^n.m.(m+1)\ldots(m+n-1).\frac{\cos(n-m)\theta}{2^n(\cos\theta)^{m+n}},$$

and by taking the difference,

$$U.(U-1)\ldots(U-n+1).\sin 2U\theta$$
$$= (-1)^n.m.(m+1)\ldots(m+n-1).\frac{\sin(n-m)\theta}{2^n(\cos\theta)^{m+n}}.$$

Let $U.(U-1).(U-2)\ldots(U-n+1)$

$$= U_n - p_1U_{n-1} + p_2U_{n-2} - \&c. \pm p_{n-1} = F(U),$$

then

$$\frac{\cos(n-m)\theta}{(\cos\theta)^{m+n}} = \frac{(-1)^n.2^n}{m.(m+1)\ldots(m+n-1)}.\left\{F(U) - \frac{U^2F(U).\theta^2}{1.2}\right.$$
$$\left.+\frac{U^4F(U)\,\theta^4}{1.2.3.4} - \&c.\right\}\ \ldots\ldots\ldots\text{(XVII.)},$$

and $\frac{\sin(n-m)\theta}{(\cos\theta)^{m+n}}$

$$= \frac{(-1)^n\,2^n}{m.(m+n)\ldots(m+n-1)}\left\{UF(U)\,\theta - \frac{U^3F(U)\,\theta^3}{1.2.3} + \&c.\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(XVIII.)},$$

where $U^r F(U) = U_{n+r} - p_1 U_{n+r-1} + p_2 U_{n+r-2} - \&c. \pm p_{n-1} U_r$, which, by (XI.) can be expressed in terms of the A, and therefore of the B numbers.

28. From preceding Art. we obtain, by multiplying by 2^n,

$$2U(2U-2)(2U-4)\ldots\{2U-(2n-2)\}\cos 2U\theta$$
$$= (-1)^n . m.(m+1)\ldots(m+n-1).\frac{\cos(n-m)\,\theta}{(\cos\theta)^{m+n}},$$
$$2U(2U-2)\ldots\{2U-(2n-2)\}\sin 2U\theta$$
$$= (-1)^n . m.(m+1)\ldots(m+n-1).\frac{\sin(n-m)\,\theta}{(\cos\theta)^{m+n}}.$$

Let $c_1, c_2, c_3, \ldots c_{n-1}$ be the sums of the products of the $(n-1)$ quantities, viz., $2, 4, 6, \ldots(2n-2)$, taken $1, 2, 3, \ldots(n-1)$ together, then

$$\{(2U)^n - c_1(2U)^{n-1} + c_2(2U)^{n-2} - \&c. \pm c_{n-1}(2U)\}\cos 2U\theta$$
$$= (-1)^n . m.(m+1)\ldots(m+n-1)\,\frac{\cos(n-m)\,\theta}{(\cos\theta)^{m+n}},$$

and $\{(2U)^n - c_1(2U)^{n-1} + \ldots \pm c_{n-1}(2U)\}\sin 2U\theta$

$$= (-1)^n\, m.(m+1)\ldots(m+n-1).\frac{\sin(n-m)\,\theta}{(\cos\theta)^{m+n}}.$$

Also $$\cos 2U\theta = \frac{\cos m\theta}{(\cos\theta)^m};$$

therefore $$-2U\sin 2U\theta = \frac{d}{d\theta}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\},$$

$$-(2U)^2\cos 2U\theta = \frac{d^2}{(d\theta)^2}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\},$$

$$(2U)^3\sin 2U\theta = \frac{d^3}{(d\theta)^3}.\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\},$$

$$(2U)^4\cos 2U\theta = \frac{d^4}{(d\theta)^4}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\} \;\&c.\ \&c.$$

Similarly, $$\sin 2U\theta = -\frac{\sin m\theta}{(\cos\theta)^m};$$

therefore $$-2U\cos 2U\theta = \frac{d}{d\theta}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\},$$

$$(2U)^2\sin 2U\theta = \frac{d^2}{(d\theta)^2}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\},$$

$$(2U)^3 \cos 2U\theta = \frac{d^3}{(d\theta)^3} \left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\},$$

$$-(2U)^4 \sin 2U\theta = \frac{d^4}{(d\theta)^4} \left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} \text{ \&c.}$$

Hence by substitution and reduction we obtain the following results:

$$\frac{d^n}{d\theta^n}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\} - c_1\frac{d^{n-1}}{d\theta^{n-1}}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\}$$
$$- c_2\frac{d^{n-2}}{d\theta^{n-2}}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\} + c_3\frac{d^{n-3}}{d\theta^{n-3}}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} + \text{\&c.}$$
$$= (-1)^{\frac{1}{2}n}.m.(m+1).(m+2)\ldots(m+n-1).\frac{\cos(n-m)\,\theta}{(\cos\theta)^{m+n}} \;(n \text{ even}) \quad\ldots\ldots(30),$$

$$\text{and} = (-1)^{\frac{1}{2}(n-1)}.m.(m+1)\ldots(m+n-1).\frac{\sin(n-m)\,\theta}{(\cos\theta)^{m+n}} \;(n \text{ odd}) \quad\ldots\ldots(31),$$

$$\frac{d^n}{d\theta^n}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} + c_1\frac{d^{n-1}}{d\theta^{n-1}}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\}$$
$$- c_2\frac{d^{n-2}}{d\theta^{n-2}}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} - c_3.\frac{d^{n-3}}{d\theta^{n-3}}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\} + \text{\&c.}$$
$$= (-1)^{\frac{1}{2}(n+1)}.m.(m+1)\ldots(m+n-1).\frac{\sin(n-m)\,\theta}{(\cos\theta)^{m+n}} \;(n \text{ even}) \quad\ldots\ldots(32),$$

$$\text{and} = (-1)^{\frac{1}{2}n-1}.m.(m+1)\ldots(m+n-1).\frac{\cos(n-m)\,\theta}{(\cos\theta)^{m+n}} \;(n \text{ odd}) \quad\ldots\ldots(33),$$

If $m = 1$, these equations become

$$-c_1\frac{d^{n-1}}{d\theta^{n-1}}(\tan\theta) + c_3\frac{d^{n-3}}{d\theta^{n-3}}(\tan\theta) - \text{\&c.}$$
$$= (-1)^{\frac{1}{2}n}\, 1.2.3\ldots n.\frac{\cos(n-1)\,\theta}{(\cos\theta)^{n+1}} \;(n \text{ even}),$$

$$\text{and} \qquad = (-1)^{\frac{1}{2}(n-1)}1.2\ldots n.\frac{\sin(n-1)\,\theta}{(\cos\theta)^{n+1}} \;(n \text{ odd}),$$

$$\frac{d^n}{d\theta^n}(\tan\theta) - c_2\frac{d^{n-2}}{d\theta^{n-2}}(\tan\theta) + \text{\&c.}$$
$$= (-1)^{\frac{1}{2}n+1}.1.2\ldots n.\frac{\sin(n-1)\,\theta}{(\cos\theta)^{n+1}} \;(n \text{ even}),$$

and $$= (-1)^{\frac{1}{2}(n-1)}.1.2\ldots n.\frac{\cos(n-1)\,\theta}{(\cos\theta)^{n+1}}\ (n \text{ odd}).$$

Ex. Let $n = 3$; therefore from (33)

$$\frac{d^3}{d\theta^3}\tan\theta - 8\,\frac{d}{d\theta}\,(\tan\theta) = -\,1.2.3\,\frac{\cos 2\theta}{(\cos\theta)^4},$$

which is the case, and from (31)

$$-\,6\,\frac{d^2}{(d\theta)^2}\,(\tan\theta) = -\,1.2.3\,\frac{\sin 2\theta}{\cos\theta^4},$$

which is also the case.

29. Many general results analogous to those of the preceding Art. may be obtained by somewhat varying the process there used. The following result, the proof of which is omitted on account of its length, is given as an instance of the power of the method adopted in this theory, and as not likely to be obtained in any other way.

Let $$N_1 = n.(n+1).(n+2)\ldots(2n-1),$$
$$N_2 = (n+1).(n+2)\ldots 2n$$
$$N_3 = (n+2).(n+3)\ldots(2n+1),\ \&c.\ \&c.,$$

and let $r_1, r_2, r_3, \ldots r_{n-1}$ be the sums of the products of the $(n-1)$ quantities, viz., $2^2, 4^2, 6^2, \ldots (2n-2)^2$, taken 1, 2, 3, ...$(n-1)$ together, then it can be shewn that

$$\frac{d^{2n}}{d\theta^{2n}}\left(\frac{\cos m\theta}{\cos\theta^m}\right) + r_1\,\frac{d^{2n-2}}{d\theta^{2n-2}}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\} + r_2\,\frac{d^{2n-4}}{d\theta^{2n-4}}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\} + \ldots$$
$$+ r_{n-1}\,\frac{d^2}{d\theta^2}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\}$$
$$= (-1)^n\, m.(m+1)\ldots(m+n-1)\left\{\frac{N_1 2^m}{(\cos\theta)^{2n}} - \frac{m}{1}.N_2.2^{m-1}.\frac{\cos\theta}{(\cos\theta)^{2n+1}}\right.$$
$$\left. + \frac{m.(m-1)}{1.2}.N_3.2^{m-2}\,\frac{\cos 2\theta}{(\cos\theta)^{2n+2}} - \&c.\right\}\ \ldots\ldots\ldots(34),$$

and $$\frac{d^{2n}}{d\theta^{2n}}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} + r_1\,\frac{d^{2n-2}}{d\theta^{2n-2}}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} + r_2\,\frac{d^{2n-4}}{d\theta^{2n-4}}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\} + \ldots$$
$$+ r_{n-1}\,\frac{d^2}{d\theta^2}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\}$$
$$= (-1)^n.m.(m+1)\ldots(m+n-1)\left\{\frac{m}{1}\,N_2.2^{m-1}\,\frac{\sin\theta}{(\cos\theta)^{2n+1}}\right.$$
$$\left. - \frac{m.(m-1)}{1.2}\,N_3\,\frac{\sin 2\theta}{(\cos\theta)^{2n+2}} + \&c.\right\}\ \ldots\ldots\ldots(35),$$

Let $m=1$, then from (35),

$$\frac{d^{2n}}{d\theta^{2n}}(\tan\theta)+r_1\frac{d^{2n-2}}{d\theta^{2n-2}}(\tan\theta)+\&c.=(-1)^n\,1.2.3\ldots 2n\,\frac{\sin\theta}{(\cos\theta)^{2n+1}}.$$

Ex. Let $n=2$; therefore

$$\frac{d^4}{d\theta^4}(\tan\theta)+4\frac{d^2}{d\theta^2}(\tan\theta)=1.2.3.4\,\frac{\sin\theta}{(\cos\theta)^5},$$

which is the case.

Cor. $\dfrac{d^{2n-1}}{d\theta^{2n-1}}(\tan\theta)+r_1\dfrac{d^{2n-3}}{d\theta^{2n-3}}(\tan\theta)+\ldots+r_{n-1}\dfrac{d}{d\theta}(\tan\theta)$

$$=(-1)^n.1.2\ldots 2n\int\frac{\sin\theta\,d\theta}{(\cos\theta)^{2n-1}}=\frac{(-1)^n.1.2\ldots(2n-1)}{(\cos\theta)^{2n}}.$$

30. In $\qquad \varepsilon^{-U\theta}=\dfrac{2^m\varepsilon^{m\theta}}{(\varepsilon^\theta+1)^m}$ put x for ε^θ;

therefore $\qquad x^{-U}=\dfrac{2^m x^m}{(x+1)^m}$;

therefore $\qquad x^{-(U+m)}=\dfrac{2^m}{(x+1)^m}$.

Hence by successive differentiation,

$(U+m)(U+m+1)\ldots(U+m+n-1)\,x^{-(U+m+n)}$

$$=2^m.m.(m+1)\ldots(m+n-1).\frac{1}{(x+1)^{m+n}};$$

therefore $(U+m)(U+m+1)\ldots(U+m+n-1)\,x^{-U}$

$$=2^m.m.(m+1)\ldots(m+n-1).\left(\frac{x}{x+1}\right)^{m+n};$$

therefore putting $x=\varepsilon^\theta$,

$(U+m)(U+m+1)\ldots(U+m+n-1)\,\varepsilon^{-U\theta}$

$$=\frac{m.(m+1)\ldots(m+n-1)}{2^n}.\frac{2^{m+n}.\varepsilon^{(m+n)\theta}}{(\varepsilon^\theta+1)^{m+n}};$$

which $\qquad =\dfrac{m.(m+1)\ldots(m+n-1)}{2^n}.\varepsilon^{-U'\theta}$,

if U' is what U becomes when $m+n$ is put for m; therefore equating coefficients of θ^r,

$(U+m)(U+m+1)\ldots(U+m+n-1)\,U^r$

$$=\frac{m.(m+1)\,(m+n-1)}{2^m}.U'_r.$$

Let $c_1, c_2, \ldots c_n$ be the sums of the products of the n quantities, viz., m, $(m+1)$, $(m+2)$, $\ldots(m+n-1)$ taken 1, 2, 3, $\ldots n$ together, then

$$U_{n+r}+c_1 U_{n+r-1}+c_2 U_{n+r-2}+\ldots+c_n U_r=\frac{m.(m+1)\ldots(m+n-1)}{2^n}.U_r'.$$

Since the U numbers involve m, this fact may be exhibited in the notation by putting ${}_mU_r$ for U_r, &c., and we have from the above equation

$${}_{m+n}U_r=\frac{2^n}{m.(m+1)\ldots(m+n-1)}\left({}_mU_{n+r}+c_1.{}_mU_{n+r-1}\right.$$
$$\left.+c_2.{}_mU_{n+r-2}+\ldots+c_n.{}_mU_r\right)\ldots\ldots\ldots\ldots\ldots(\text{XIX.}).$$

When $m=1$ put $n=m-1$, and we have

$${}_mU_r=\frac{2^{m-1}}{1.2\ldots(m-1)}(A_{m+r-1}+c_1A_{m+r-2}+\ldots+c_{m-1}A_r),$$

which is the formula (XI.), of which therefore the formula (XIX.) is a generalization.

31. Required to express U_n and consequently A_n in direct terms.

Since $x^U=\dfrac{2^m}{(x+1)^m}$; therefore by differentiation,

$$Ux^{U-1}=-\frac{m2^m}{(x+1)^{m+1}};$$

therefore $Ux^U=-\dfrac{m2^m x}{(x+1)^{m+1}}=-m.2^m\left\{\dfrac{1}{(x+1)^m}-\dfrac{1}{(x+1)^{m+1}}\right\}.$

Similarly by a second differentiation and reduction, we obtain

$$U^2x^U=2^m\left[\frac{m^2}{(x+1)^m}-\frac{m}{1}\left\{\frac{(m+1)^2-m^2}{(x+1)^{m+1}}\right\}\right.$$
$$\left.+\frac{m.(m+1)}{1.2}\left\{\frac{(m+2)^2-2.(m+1)^2+m^2}{(x+1)^{m+2}}\right\}\right],$$

repeating the process, we have

$$U^3x^U=-2^m\left[\frac{m^3}{(x+1)^m}-\frac{m}{1}\left\{\frac{(m+1)^3-m^3}{(x+1)^{m+1}}\right\}\right.$$
$$+\frac{m.(m+1)}{1.2}\left\{\frac{(m+2)^3-2.(m+1)^3+m^3}{(x+1)^{m+2}}\right\}$$
$$\left.-\frac{m.(m+1).(m+2)}{1.2.3}\left\{\frac{(m+3)^3-3.(m+2)^3+3.(m+1)^3-m^3}{(x+1)^{m+3}}\right\}\right].$$

Hence generally, the law of formation of terms being evident,

$$U^n x^U = (-1)^n\, 2^m \left[\frac{m^n}{(x+1)^m} - \frac{m}{1} \cdot \left\{\frac{(m+1)^n - m^n}{(x+1)^{m+1}}\right\}\right.$$

$$\left. + \frac{m.(m+1)}{1.2} \left\{\frac{(m+2)^n - 2.(m+1)^n + m^n}{(x+1)^{m+2}}\right\} - \&c.\right] \text{....(XX.)}.$$

Let $x=1$; therefore

$$U_n = (-1)^n \left[m^n - \frac{m}{1} \cdot \left\{\frac{(m+1)^n - m^n}{2}\right\}\right.$$

$$\left. + \frac{m.(m+1)}{1.2} \left\{\frac{(m+2)^n - 2.(m+1)^n + m^n}{2^2}\right\} - \&c.\right] \text{....... (36).}$$

Let $m=1$; therefore

$$A_n = (-1)^n \left(1^n - \frac{2^n - 1^n}{2} + \frac{3^n - 2.2^n + 1^n}{2^2} - \&c.\right) \text{......... (37).}$$

Cor. In (XX.) put for x, $\varepsilon^{2\theta\sqrt{(-1)}}$, $\varepsilon^{-2\theta\sqrt{(-1)}}$ successively, and multiply by 2^n, then

$$2^n U^n \cos 2U\theta = (-1)^n\, 2^n \left\{m^n \frac{\cos m\theta}{(\cos\theta)^m}\right.$$

$$- \frac{m}{1} \cdot \frac{(m+1)^n - m^n}{2} \cdot \frac{\cos(m+1)\,\theta}{(\cos\theta)^{m+1}}$$

$$\left. + \frac{m.(m+1)}{1.2} \cdot \frac{(m+2)^n - 2.(m+1)^n + m^n}{2^2} \cdot \frac{\cos(m+2)\,\theta}{(\cos\theta)^{m+2}} - \&c.\right\},$$

$$2^n U^n \sin 2U\theta = (-1)^{n+1}\, 2^n \left\{m^n \frac{\sin m\theta}{(\cos\theta)^m}\right.$$

$$- \frac{m}{1} \cdot \frac{(m+1)^n - m^n}{2} \cdot \frac{\sin(m+1)\,\theta}{(\cos\theta)^{m+1}}$$

$$\left. + \frac{m.(m+1)}{1.2} \cdot \frac{(m+2)^n - 2.(m+1)^n + m^n}{2^2} \cdot \frac{\sin(m+2)\,\theta}{(\cos\theta)^{m+2}} - \&c.\right\}.$$

But $\quad 2^n U^n \cos 2U\theta = (-1)^{\frac{1}{2}n} \dfrac{d^n}{(d\theta)^n} \cdot \left\{\dfrac{\cos m\theta}{(\cos\theta)^m}\right\}$ (n even),

and $\quad = (-1)^{\frac{1}{2}(n+1)} \dfrac{d^n}{d\theta^n} \left(\dfrac{\sin m\theta}{\cos\theta^m}\right)$ (n odd),

and $\quad 2^n U^n \sin 2U\theta = (-1)^{\frac{1}{2}n} \dfrac{d^n}{(d\theta)^n} \left\{\dfrac{\sin m\theta}{(\cos\theta)^m}\right\}$ n even,

and $\quad = (-1)^{\frac{1}{2}(n+1)} \dfrac{d^n}{d\theta^n} \left\{\dfrac{\cos m\theta}{(\cos\theta)^m}\right\}$ (n odd);

therefore $\frac{d^n}{d\theta^n}\left\{\frac{\cos m\theta}{(\cos\theta)^m}\right\}=(-1)^{\frac{1}{2}n}.2^n\left\{m^n\frac{\cos m\theta}{(\cos\theta)^m}-\&c.\right\}$

n even..............(38),

and $\qquad =(-1)^{\frac{1}{2}(n+1)}.2^n\left\{m^n\frac{\sin m\theta}{(\cos\theta)^m}-\&c.\right\}$ n odd......(39),

also $\frac{d^n}{(d\theta)^n}\left\{\frac{\sin m\theta}{(\cos\theta)^m}\right\}=(-1)^{\frac{1}{2}n}.2^n\left\{m^n\frac{\sin m\theta}{(\cos\theta)^m}-\&c.\right\}$

n even..............(40),

and $\qquad =(-1)^{\frac{1}{2}(n-1)}2^n\left\{m^n\frac{\cos m\theta}{(\cos\theta)^m}-\&c.\right\}$ n odd ...... (41).

COR. Let $m=1$; therefore

$$\frac{d^n}{d\theta^n}(\tan\theta)=(-1)^{\frac{1}{2}n}.2^n\left\{\frac{\sin\theta}{\cos\theta}-\frac{2^n-1}{2}\cdot\frac{\sin 2\theta}{(\cos\theta)^2}\right.$$
$$\left.+\frac{3^n-2.2^n+1}{2^2}\cdot\frac{\sin 3\theta}{(\cos\theta)^3}-\&c.\right\}\ n \text{ even},$$

$$\text{and }=(-1)^{\frac{1}{2}(n-1)}.2^n\left\{1-\frac{2^n-1}{2}\cdot\frac{\cos 2\theta}{(\cos\theta)^2}\right.$$
$$\left.+\frac{3^n-2.2^n+1}{2^2}\cdot\frac{\cos 3\theta}{(\cos\theta)^3}-\&c.\right\}\ n \text{ odd}.$$

EX. Let $x=2$; therefore

$$\frac{d^2}{d\theta^2}(\tan\theta)=-4\left\{\frac{\sin\theta}{\cos\theta}-\tfrac{3}{2}.\frac{\sin 2\theta}{\cos^2\theta}+\tfrac{1}{2}\frac{\sin 3\theta}{(\cos\theta)^3}\right\}=\frac{2\sin\theta}{(\cos\theta)^3}.$$

32. Required to express B_n in direct terms.

Since

$$(1+x)^B=\frac{\log(1+x)}{x}=1-\frac{x}{2}+\frac{x^2}{3}-\&c.+(-1)^r\frac{x^r}{r+1}+\&c.,$$

hence by differentiation and reduction we obtain

$$B(1+x)^B=\Sigma\left\{\left(-\frac{1}{r+1}+\frac{1}{r+2}\right)(-x)^r\right\}\ \begin{matrix}r=0\\ r=\infty\end{matrix}.$$

Similarly by repeating the process,

$$B^2(1+x)^B=\Sigma\left\{\left(\frac{1}{r+1}-\frac{3}{r+2}+\frac{2}{r+3}\right)(-x)^r\right\},$$

$$B^3(1+x)^B=\Sigma\left\{\left(-\frac{1^3}{r+1}+\frac{2^3-1^3}{r+2}-\frac{3^3-2.2^3+1^3}{r+3}\right.\right.$$
$$\left.\left.+\frac{4^3-3.3^3+3.2^3-1^3}{r+4}\right)(-x)^r\right\}.$$

Hence generally, the law of the formation of terms being evident,

$$B^n(1+x)^B=(-1)^n\left\{\Sigma\left(\frac{1^n}{r+1}-\frac{2^n-1^n}{r+2}+\frac{3^n-2.2^n+1^n}{r+3}\right.\right.$$
$$\left.\left.-\frac{4^n-3.3^n+3.2^n-1^n}{r+4}+\&c.\right)(-x)^r\right\}\begin{matrix}r=0\\ r=\infty\end{matrix}.$$

Let $x=0$; therefore $x^r=0$ ($r>0$), and

$$B_n=(-1)^n\left(\frac{1^n}{1}-\frac{2^n-1^n}{2}+\frac{3^n-2.2^n+1^n}{3}-\&c.\ \&c.\right)\ldots(42).$$

COR. Since $A_n=-\dfrac{2^{n+2}-2}{n+1}B_{n+1}$; therefore from (37) and (42) we obtain

$$\frac{2^{n+2}-2}{n+1}\left(1-\frac{2^{n+1}-1}{2}+\frac{3^{n+1}-2.2^{n+1}+1}{3}-\&c.\right)=1-\frac{2^n-1}{2}$$
$$+\frac{3^n-2.2^n+1}{2^2}-\frac{4^n-3.3^n+3.2^n-1^n}{2^3}+\&c.\ \ldots\ldots\ldots(43).$$

This equation holds generally, *i.e.* whatever value be given to n.

33. From $x^B=\dfrac{\log x}{x-1}$ may be obtained, by successive differentiation and reduction,

$$B^nx^B=(-1)^{n+1}n\left\{\frac{1}{x-1}+\frac{2^{n-1}-1}{(x-1)^2}+\frac{3^{n-1}-2.2^{n-1}+1}{(x-1)^3}+\&c.\right\}$$
$$+(-1)^n\log x\left\{\frac{1}{x-1}+\frac{2^n-1}{(x-1)^2}+\frac{3^n-2.2^n+1}{(x-1)^3}+\&c.\right\},$$

for x put $\varepsilon^{2\theta\sqrt{(-1)}}$, $\varepsilon^{-2\theta\sqrt{(-1)}}$ successively, then multiplying by 2^n, we get

$$2^nB^n\cos 2B\theta=(-1)^n n2^n\left\{\frac{1}{2}+\frac{2^{n-1}-1}{2^2}\cdot\frac{\cos 2\theta}{(\sin\theta)^2}\right.$$
$$\left.-\frac{3^{n-1}-2.2^{n-1}+1}{2^3}\cdot\frac{\sin 3\theta}{(\sin\theta)^3}-\frac{4^{n-1}-3.3^{n-1}+3.2^{n-1}-1}{2^4}\cdot\frac{\cos 4\theta}{(\sin\theta)^4}+\&c.\right\}$$
$$+(-1)^n2^n\theta\left\{\frac{\cos\theta}{2\sin\theta}-\frac{2^n-1}{2^2}\cdot\frac{\sin 2\theta}{(\sin\theta)^2}-\frac{3^n-2.2^n+1}{2^3}\cdot\frac{\cos 3\theta}{(\sin\theta)^3}\right.$$
$$\left.+\frac{4^n-3.3^n+3.2^n-1}{2^4}\cdot\frac{\sin 4\theta}{(\sin\theta)^4}+\&c.\right\},$$

$$2^nB^n \sin 2B\theta = (-1)^n\, 2^n.n\left\{\tfrac{1}{2}\frac{\cos\theta}{\sin\theta} - \frac{2^{n-1}-1}{2^2}\cdot\frac{\sin 2\theta}{(\sin\theta)^2}\right.$$
$$\left. - \frac{3^{n-1}-2.2^{n-1}+1}{2^3}\,\frac{\cos 3\theta}{(\sin\theta)^3} + \&c.\right\}$$
$$+(-1)^n.2^n\theta\left\{\tfrac{1}{2} + \frac{2^n-1}{2^2}\cdot\frac{\cos 2\theta}{(\sin\theta)^2} - \frac{3^n-2.2^n+1}{2^3}\,\frac{\sin 3\theta}{(\sin\theta)^3} - \&c.\right\},$$

also $\qquad 2^nB^n \cos 2B\theta = (-1)^{\frac{1}{2}n}\dfrac{d^n}{d\theta^n}(\theta\cot\theta)\ (n \text{ even}),$

$$\text{and } = 0 \text{ when } n \text{ is odd},$$
$$2^nB^n \sin 2B\theta = 0\ (n \text{ even}),$$
$$\text{and } = (-1)^{\frac{1}{2}(n+1)}\frac{d^n}{d\theta^n}(\theta\cot\theta)\ (n \text{ odd}).$$

Hence, when n is even,

$$\frac{d^n}{(d\theta)^n}(\theta\cot\theta) = (-1)^{\frac{1}{2}n}\, n.2^n\left\{\tfrac{1}{2} + \frac{2^{n-1}-1}{2^2}\,\frac{\cos 2\theta}{(\sin\theta)^2}\right.$$
$$\left. - \frac{3^{n-1}-2.2^{n-1}+1}{2^3}\cdot\frac{\sin 3\theta}{(\sin\theta)^3} - \&c.\right\}$$
$$+(-1)^{\frac{1}{2}n}.2^{n+1}\theta\left\{\frac{\cos\theta}{2\sin\theta} - \frac{2^n-1}{2^2}\,\frac{\sin 2\theta}{(\sin\theta)^2}\right.$$
$$\left. - \frac{3^n-2.2^n+1}{2^3}\cdot\frac{\cos 3\theta}{(\sin\theta)^3} + \&c.\right\}\ldots\ldots\ldots\ldots(44),$$

and when n is odd,

$$\frac{d^n}{d\theta^n}(\theta\cot\theta) = (-1)^{\frac{1}{2}(n-1)} n.2^n\left\{\frac{\cos\theta}{2\sin\theta} - \frac{2^{n-1}-1}{2^2}\cdot\frac{\sin 2\theta}{(\sin\theta)^2}\right.$$
$$\left. - \frac{3^{n-1}-2.2^{n-1}+1}{2^3}\cdot\frac{\cos 3\theta}{(\sin\theta)^3} + \&c.\right\}$$
$$+(-1)^{\frac{1}{2}(n-1)}\, 2^{n+1}\theta\left\{\tfrac{1}{2} + \frac{2^n-1}{2^2}\cdot\frac{\cos 2\theta}{(\sin\theta)^2}\right.$$
$$\left. - \frac{3^n-2.2^n+1}{2^3}\,\frac{\sin 3\theta}{(\sin\theta)^3} - \&c.\right\}\ldots\ldots\ldots\ldots(45),$$

also $\;0 = \tfrac{1}{2} + \dfrac{2^n-1}{2^2}\cdot\dfrac{\cos 2\theta}{(\sin\theta)^2} - \dfrac{3^n-2.2^n+1}{2^3}\cdot\dfrac{\sin 3\theta}{(\sin\theta)^3} - \&c.$

$$(n \text{ even})\ \ldots\ldots\ldots\ldots\ldots(46),$$

$$=\frac{1}{2}\frac{\cos\theta}{\sin\theta}-\frac{2^n-1}{2^2}\cdot\frac{\sin 2\theta}{(\sin\theta)^2}-\frac{3^n-2.2^n+1}{2^3}\cdot\frac{\cos 3\theta}{(\sin\theta)^3}+\&c.$$

$(n \text{ odd})$ .... (47).

Ex. Let $n=2$; therefore

$$\frac{d^2}{d\theta^2}(\theta\cot\theta)=-8\left(\frac{1}{2}+\frac{1}{2^2}\cdot\frac{\cos 2\theta}{\sin^2\theta}\right)$$

$$-8\theta\left\{\frac{\cos\theta}{2\sin\theta}-\frac{3}{4}\frac{\sin 2\theta}{(\sin\theta)^2}-\frac{1}{4}\frac{\cos 3\theta}{(\sin\theta)^3}\right\}=\frac{2}{(\sin\theta)^2}-\frac{2\theta\cos\theta}{\sin^3\theta},$$

which is the case.

Also $$0=\frac{1}{2}+\frac{3}{4}\frac{\cos 2\theta}{(\sin\theta)^2}-\frac{1}{4}\frac{\sin 3\theta}{(\sin\theta)^3},$$

which is the case.

34. Required to prove Formula (XII.), viz. that

$$V_n=(-1)^{m+1}.\frac{n.(n-1).(n-2)\dots(n-m+1)}{1.2.3\dots(m-1)}\left(\frac{B_n}{n}+q_1\frac{B_{n-1}}{n-1}\right.$$

$$\left.+q_2\frac{B_{n-2}}{n-2}+\dots+q_{m-1}B_{n-m+1}\right),$$

where $q_1, q_2, \dots q_{m-1}$ are the sums of the products of the $m-1$ quantities, viz. $1, 2, \dots(m-1)$ taken $1, 2, \dots(m-1)$ together, *i.e.*

$$(x+1)(x+2)\dots(x+m-1)=x^{m-1}+q_1x^{m-2}+q_2x^{m-3}+\dots+q_{m-1}.$$

It was observed (Art. 24) that this formula could not apparently be obtained in a manner similar to that in which the corresponding formula for U_n was derived. The following is an elaborate proof of the above formula. An easier one may possibly yet be found.

In (IV.), Art. 10, viz.

$$f\{x+(V+m)\theta\}-\frac{m}{1}f\{x+(V+m-1)\theta\}$$

$$+\frac{m.(m-1)}{1.2}f\{x+(V+m-2\theta)\}-\&c.=\theta^m\frac{d^mfx}{dx^m},$$

let $\theta=1$ and $fx=x^{-n}$, then reversing the order of the series,

$$\frac{1}{(V+x)^n}-\frac{m}{1}\cdot\frac{1}{(V+x+1)^n}+\frac{m.(m-1)}{1.2}\cdot\frac{1}{(V+x+2)^n}-\&c.$$

$$=(-1)^m\frac{d^m(x^{-n})}{dx^m}=n.(n+1)\dots(n+m-1).\frac{1}{x^{m+n}}.$$

Hence, summing between $x=x$ and $x=\infty$, m times in succession, we have

$$\frac{1}{(V+x)^n}-\frac{m-1}{1}\frac{1}{(V+x+1)^n}+\frac{(m-1).(m-2)}{1.2}\cdot\frac{1}{(V+x+2)^n}-\&c.$$

$$=n.(n+1)...(n+m-1)\left\{\frac{1}{x^{m+n}}+\frac{1}{(x+1)^{m+n}}+\frac{1}{(x+2)^{m+n}}+\&c.\right\},$$

$$\frac{1}{(V+x)^n}-\frac{m-2}{1}\cdot\frac{1}{(V+x+1)^n}+\frac{(m-2).(m-3)}{1.2}\frac{1}{(V+x+2)^n}-\&c.$$

$$=n.(n+1)...(n+m-1)\left\{\frac{1}{x^{m+n}}+\frac{2}{(x+1)^{m+n}}+\frac{3}{(x+2)^{m+n}}+\&c.\right\},$$

$$\frac{1}{(V+x)^n}-\frac{m-3}{1}\cdot\frac{1}{(V+x+1)^n}+\frac{(m-3).(m-4)}{1.2}\cdot\frac{1}{(V+x+2)^n}-\&c.$$

$$=\frac{n.(n+1)...(n+m-1)}{1.2}\left\{\frac{1.2}{x^{m+n}}+\frac{2.3}{(x+1)^{m+n}}+\frac{3.4}{(x+2)^{m+n}}+\&c.\right\},$$

$$\vdots$$

$$\frac{1}{(V+x)^n}=\frac{n.(n+1)...(n+m-1)}{1.2...(m-1)}\left\{\frac{1.2...(m-1)}{x^{m+n}}\right.$$

$$\left.+\frac{2.3...m}{(x+1)^{m+n}}+\frac{3.4...(m+1)}{(x+2)^{m+n}}+\&c.\right\}.$$

Now let $x=1-1=0$, then

$$\frac{1}{V^n}(=V_{-n})=\frac{n.(n+1)...(n+m-1)}{1.2...(m-1)}\left\{\frac{1.2...(m-1)}{(1-1)^{m+n}}\right.$$

$$\left.+\frac{2.3...m}{1^{m+n}}+\frac{3.4...(m+1)}{2^{m+n}}+\&c.\right\}.$$

Again,

$$\frac{(x+1)(x+2)...(x+m-1)}{x^{m+n}}=\frac{x^{m-1}+q_1x^{m-2}+q_2x^{m-3}+...+q_{m-1}}{x^{m+n}}$$

$$=\frac{1}{x^{n+1}}+\frac{q_1}{x^{n+2}}+\frac{q_2}{x^{n+3}}+...+\frac{q_{m-1}}{x^{n+m}};$$

therefore summing between $x=0=1-1$ and $x=\infty$, we have

$$\frac{1.2...(m-1)}{(1-1)^{m+n}}+\frac{2.3...m}{1^{m+n}}+\frac{3.4...(m+1)}{2^{m+n}}+\&c.=\frac{1}{(1-1)^{n+1}}+S_{n+1}$$

$$+q_1\left\{\frac{1}{(1-1)^{n+2}}+S_{n+2}\right\}+q_2\left\{\frac{1}{(1-1)^{n+3}}+S_{n+3}\right\}+...$$

$$+q_{m-1}\left\{\frac{1}{(1-1)^{m+n}}+S_{m+n}\right\};$$

therefore

$$V_{-n} = \frac{n.(n+1)\ldots(n+m-1)}{1.2\ldots(m-1)} \left[\frac{1}{(1-1)^{n+1}} + S_{n+1}\right.$$

$$\left. + q_1 \left\{\frac{1}{(1-1)^{n+2}} + S_{n+2}\right\} + \ldots + q_{m-1} \left\{\frac{1}{(1-1)^{m+n}} + S_{m+n}\right\}\right].$$

When $m=1$ V becomes B; therefore

$$B_{-n} = \frac{n}{\Gamma(1)} \left\{\frac{1}{(1-1)^{n+1}} + S_{n+1}\right\}, \text{ but } \Gamma(1) = 1;$$

therefore $$\frac{1}{(1-1)^{n+1}} + S_{n+1} = \frac{B_{-n}}{n},$$

Hence $$V_{-n} = \frac{(n.(n+1)\ldots(n+m-1)}{1.2\ldots(m-1)} \left\{\frac{B_{-n}}{n} + q_1 \frac{B_{-(n+1)}}{n+1}\right.$$

$$\left. + q_2 \frac{B_{-(n+2)}}{n+2} + \ldots + q_{m-1} \frac{B_{-(m+n-1)}}{m+n-1}\right\}.$$

Now for n put $-n$, and

$$V_n = (-1)^{m+1}.\frac{n.(n-1)\ldots(n-m+1)}{1.2\ldots(m-1)} \left(\frac{B_n}{n} + q_1 \frac{B_{n-1}}{n-1}\right.$$

$$\left. + q_2 \frac{B_{n-2}}{n-2} + \ldots + q_{m-1} \frac{B_{n-m+1}}{n-m+1}\right).$$

COR.* If $n < m$, then

$$V_n = (-1)^{m+1}.\frac{n.(n-1)\ldots 2.1.0 \times (-1)^{m-n-1}.1.2\ldots(m-n-1)}{1.2\ldots(m-1)} \left\{\frac{B_n}{n}\right.$$

$$\left. + q_1 \frac{B_{n-1}}{n-1} + \ldots + q_n \frac{B_0}{0}\right\},$$

* The generic equation

$$(V+m)^n - m(V+m-1)^n + \frac{m.(m-1)}{1.2}(V+m-2)^n - \&c. = 0,$$

will give, for the determination of the V numbers, the following equation expressed in the received notation, viz.

$$\Delta^m 0^{m+n} + \frac{m+n}{1}\Delta^m 0^{m+n-1}.V_1 + \frac{(m+n).(m+n-1)}{1.2}\Delta^m 0^{m+n-2}.V_2 + \ldots$$

$$+ \frac{(m+n).(m+n-1)\ldots(m+1)}{1.2\ldots n}\Delta^m 0^m.V_n = 0.$$

This equation, by above Cor., becomes ($n < m$)

$$\Delta^m 0^{m+n} - \frac{m+n}{m-1} q_1 \Delta^m 0^{m+n-1} + \frac{(m+n).(m+n-1)}{(m-1).(m-2)} q_2 \Delta^m 0^{m+n-2} - \&c.$$

$$+ (-1)^n \frac{(m+n).(m+n-1)\ldots(m+1)}{(m-1).(m-2)\ldots(m-n)} q_n \Delta^m 0^m = 0.$$

which is reduced to the last term, and becomes (since $B_0 = 1$),

$$V_n\,(n<m) = (-1)^n.\frac{n.(n-1)\ldots2.1}{(m-1).(m-2)\ldots(m-n)}.q_n\ldots\ldots(\text{XXI.}),$$

a remarkable property of numbers which is probably new.

Ex. 1. Let $m=4$; therefore $q_1 = 6$, $q_2 = 11$, $q_3 = 6$. Hence $V_1 = -\frac{1}{3}q_1 = -2$, $V_2 = \frac{2.1}{3.2}\,q_2 = \frac{11}{3}$, $V_3 = \frac{3.2.1}{3.2.1}.q_3 = -6$.

Again, from

$$(V+4)^n - 4\,(V+3)^n + 6\,(V+2)^n - 4\,(V+1)^n + V^n = 0,$$

by putting for n the quantities 5, 6, 7 successively, we obtain, upon reduction,

$$V_1 + 2 = 0,\ 3\,V_2 + 12\,V_1 + 13 = 0,\ V_3 + 6\,V_2 + 13\,V_1 + 10 = 0,$$

which give the above values, viz.

$$V_1 = -2,\ V_2 = \tfrac{11}{3},\ V_3 = -6.$$

Ex. 2. In Art. 25, we have $(V + \frac{1}{2}m)^{2n+1} = 0$. Let $m=6$, $n=1$; therefore $(V+3)^3 = 0$; therefore

$$3^3 + 3.3^2\,V_1 + 3.3\,V_2 + V_3 = 0,$$

but from (XXI.),

$$V_1 = -\tfrac{1}{5}q_1,\ V_2 = \frac{2.1}{5.4}\,q_2,\ V_3 = -\frac{3.2.1}{5.4.3}\,q_3,\ \&c.,$$

where q_1, q_2, q_3, &c. are the sums of the products of the quantities 1, 2, 3, 4, 5 taken 1, 2, 3, &c. together; therefore

$$q_1 = 15,\ q_2 = 85,\ q_3 = 225;$$

therefore $27 + 27\,V_1 + 9\,V_2 + V_3 = 27 - \frac{27}{5}.15 + \frac{9}{10}.85 - \frac{225}{10}$
$= 27 - 81 + \frac{153}{2} - \frac{45}{2} = 0.$

35. The preceding developments have all been derived from two classes of generic equations, viz. those which determine the U and V numbers, of which the A and B numbers are particular cases arising from putting $m=1$.

The number of generic equations, however, which may be assumed is unlimited. The following instance is given as shewing the *working power* of the method and notation employed in this theory.

Let the following generic equation be assumed, viz.

$$nP_{n-1} + \frac{n.(n-1)}{1.2}\,P_{n-2} + \frac{n.(n-1).(n-2)}{1.2.3}\,P_{n-3} + \ldots + nP_1 + 1 = B_n,$$

where B is the representative of Bernoulli's numbers, n being a positive integer > 1. It is required to determine the values of the P numbers, *i.e.*, $P_1, P_2, \ldots P_n$. Giving to n the successive values of 2, 3, 4, &c., we have

$$2P_1 + 1 = B_2,\ 3P_2 + 3P_1 + 1 = B_3,\ \text{\&c.}$$

Hence $\quad P_1 = -\frac{5}{12},\ P_2 = \frac{1}{12},\ P_3 = \frac{1}{80},\ P_4 = -\frac{1}{80},$ &c.

The value of P_n cannot by this method be expressed in general terms, but may be elegantly obtained as follows:

By representative notation, the above generic equation becomes

$$(P+1)^n - P^n = B_n = B^n\ (n > 1)\ ;$$

therefore $\quad f\{x + (P+1)\,\theta\} - f(x + P\theta),$

$$\text{which} = \frac{dfx}{dx}\cdot\frac{\theta}{1} + \frac{d^2fx}{dx^2}\{(P+1)^2 - P^2\}\frac{\theta^2}{1.2}$$

$$+ \frac{d^3fx}{dx^3}\{(P+1)^3 - P^3\}\frac{\theta^3}{1.2.3} + \text{\&c.}$$

$$= \frac{dfx}{dx}\cdot\frac{\theta}{1} + \frac{d^2fx}{dx^2}\cdot\frac{B^2\theta^2}{1.2} + \frac{d^3fx}{dx^3}\cdot\frac{B^3\theta^3}{1.2.3} + \text{\&c.}$$

$$= f(x + B\theta) - f(x) + (1 - B_1)\,\theta\,\frac{dfx}{dx}$$

$$= f(x + B\theta) - f(x) + \tfrac{1}{2}3\theta\,\frac{dfx}{dx}\ \left(\text{since } B_1 = -\tfrac{1}{2}\right).$$

Let $fx = \varepsilon^x$, then $(x = 0)\ \varepsilon^{(P+1)\theta} - \varepsilon^{P\theta} = \varepsilon^{B\theta} - 1 + \frac{1}{2}3\theta$; therefore

$$\varepsilon^{P\theta}(\varepsilon^\theta - 1) = \varepsilon^{B\theta} - 1 + \tfrac{1}{2}3\theta\,;$$

therefore $\quad \varepsilon^{P\theta} = \dfrac{\varepsilon^{B\theta}}{\varepsilon^\theta - 1} - \dfrac{1}{\varepsilon^\theta - 1} + \dfrac{3}{2}\,\dfrac{\theta}{\varepsilon^\theta - 1}.$

But $\quad \varepsilon^{B\theta} = \dfrac{\theta}{\varepsilon^\theta - 1}\,;$

therefore $\quad \varepsilon^{P\theta} = \dfrac{1}{\theta}\cdot\dfrac{\theta^2}{(\varepsilon^\theta - 1)^2} - \dfrac{1}{\theta}\,\varepsilon^{B\theta} + \tfrac{3}{2}\varepsilon^{B\theta}.$

Again from (IV.), Art. 10, if $m = 2$,

$$\varepsilon^{V\theta} = \frac{\theta^2}{(\varepsilon^\theta - 1)^2},$$

therefore $\quad \varepsilon^{P\theta} = \dfrac{1}{\theta}\,\varepsilon^{V\theta} - \dfrac{1}{\theta}\,\varepsilon^{B\theta} + \tfrac{3}{2}\varepsilon^{B\theta}.$

Equating coefficients of θ^n, we have

$$P_n = \frac{1}{n+1}\,V_{n+1} - \frac{1}{n+1}\,B_{n+1} + \tfrac{1}{2}B_n.$$

Also from (XII.), when $m = 2$,

$$V_{n+1} = -(n+1).n\left(\frac{B_{n+1}}{n+1} + \frac{B_n}{n}\right) = -nB_{n+1} - (n+1)\,B_n;$$

therefore $$P_n = \tfrac{1}{2}B_n - B_{n+1}.$$

Hence $P_1 = -\frac{1}{4} - \frac{1}{6} = -\frac{5}{12}$, $P_2 = \frac{1}{12}$, $P_3 = -B_4 = \frac{1}{30}$, &c.,

$$P_{2n} = \tfrac{1}{2}B_{2n},\ P_{2n+1} = -B_{2n+2}\ (n > 0).$$

Scholium. The subject of generic equations appears to furnish a mine of analytical research, in which, by aid of my notation, as a fitting tool to work with, whatever direction may be taken, new and highly general results appear capable of being turned up in considerable abundance. Besides developments, the method has been applied to transcendents, generalization, and series, under each of which heads various results can be exhibited of undoubted novelty, elegance, and generality. It is, I think, evident that the notation here used is alike simple and effective, and possesses large generalizing power, as shewn both by the nature of the results obtained and by the facility of their production. On these grounds I trust that the mathematical reader will kindly bear with my introducing to his notice a new notation, new at least, as regards the use here made of it, and not consider me justly chargeable with a needless and presumptuous innovation.

The *divided notation* of the calculus of operations, from being based on a separation of symbols, would, as I conceive, if substituted for my own in the present theory, prove only an encumbrance and a disadvantage.

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Newbury, Berks.

(*To be continued.*)

ON THE VARIATIONS OF THE NODE AND INCLINATION IN THE PLANETARY THEORY.

By C. H. H. Cheyne, B.A., St. John's College.

The following method of obtaining the variation of the inclination, which differs from that which I have given in the last number of this *Journal*, is rigorous, and possesses in addition the merit of brevity.

The same notation will be employed, and we shall denote by ϑ the longitude of the disturbed planet, measured on the plane of reference as far as the node, and thence on the plane of the orbit.

Lemma. To shew that $\dfrac{dR}{d\theta_1} = \dfrac{dR}{d\vartheta} + \dfrac{dR}{d\gamma_1}$. We have

$$\sin\lambda = \sin i_1 \sin(\vartheta - \gamma_1),$$

$$r_1 = r\cos\lambda,$$

$$\tan(\theta_1 - \gamma_1) = \cos i_1 \tan(\vartheta - \gamma_1),$$

$$z = r\sin\lambda,$$

whence

$$r_1 = \phi(\vartheta - \gamma_1),$$

$$\theta_1 - \gamma_1 = \chi(\vartheta - \gamma_1),$$

$$z = \psi(\vartheta - \gamma_1),$$

where ϕ, χ, ψ are symbols of functionality.

It follows that

$$\frac{dr_1}{d\vartheta} + \frac{dr_1}{d\gamma_1} = 0,$$

$$\frac{d\theta_1}{d\vartheta} + \frac{d\theta_1}{d\gamma_1} = 1,$$

$$\frac{dz}{d\vartheta} + \frac{dz}{d\gamma_1} = 0.$$

Now

$$\frac{dR}{d\vartheta} = \frac{dR}{dr_1}\frac{dr_1}{d\vartheta} + \frac{dR}{d\theta_1}\frac{d\theta_1}{d\vartheta} + \frac{dR}{dz}\frac{dz}{d\vartheta},$$

$$\frac{dR}{d\gamma_1} = \frac{dR}{dr_1}\frac{dr_1}{d\gamma_1} + \frac{dR}{d\theta_1}\frac{d\theta_1}{d\gamma_1} + \frac{dR}{dz}\frac{dz}{d\gamma_1};$$

therefore, by addition,

$$\frac{dR}{d\vartheta} + \frac{dR}{d\gamma_1} = \frac{dR}{d\theta_1}.$$

COR. Since $\dfrac{dR}{d\vartheta} = \dfrac{dR}{d\epsilon_1} + \dfrac{dR}{d\varpi_1}$,

this may be written

$$\frac{dR}{d\theta_1} = \frac{dR}{d\epsilon_1} + \frac{dR}{d\varpi_1} + \frac{dR}{d\gamma_1}.$$

We are now in a position to obtain the required variation. For, differentiating the formula

$$H = h_1 \cos i_1,$$

$$\frac{dH}{dt} = \cos i_1 \frac{dh_1}{dt} - h_1 \sin i_1 \frac{di_1}{dt},$$

and by the equations of motion,

$$\frac{dH}{dt} = \frac{dR}{d\theta_1}, \quad \frac{dh_1}{dt} = \frac{dR}{d\theta} * = \frac{dR}{d\vartheta};$$

therefore $-h_1 \sin i_1 \dfrac{di_1}{dt} = \dfrac{dR}{d\theta_1} - \cos i_1 \dfrac{dR}{d\vartheta}$

$$= \frac{dR}{d\gamma_1} + (1 - \cos i_1) \frac{dR}{d\vartheta}, \quad (\textit{Lemma})$$

$$= \frac{dR}{d\gamma_1} + (1 - \cos i_1) \left(\frac{dR}{d\epsilon_1} + \frac{dR}{d\varpi_1}\right).$$

When either the variation of the node, or that of the inclination has been obtained, the other may be thus deduced.

The function R may be expressed in two ways,

$$R = f(r, \theta, i_1, \gamma_1, \nu), \text{ or } = \phi(r, \vartheta, i_1, \gamma_1),$$

which are derivable the one from the other by the relation

$$\vartheta = \theta + \gamma_1 - \nu \text{ ....} (1).$$

Differentiating the former as if the elements were invariable,

$$\frac{dR}{dt} = \frac{df}{dr}\frac{dr}{dt} + \frac{df}{d\theta}\frac{d\theta}{dt} \text{..................} (2),$$

* If the acceleration on the planet be resolved in the plane of its orbit along and perpendicular to its radius vector, the accelerations in these directions take the same forms as if this plane were at rest, provided θ be measured from an origin remaining fixed relatively to it.

from the latter,

$$\frac{dR}{dt} = \frac{d\phi}{dr}\frac{dr}{dt} + \frac{d\phi}{d\vartheta}\frac{d\vartheta}{dt} + \frac{d\phi}{di_1}\frac{di_1}{dt} + \frac{d\phi}{d\gamma_1}\frac{d\gamma_1}{dt} \ldots\ldots (3),$$

and from (1),

$$\frac{d\vartheta}{dt} = \frac{d\theta}{dt} + (1-\cos i_1)\frac{d\gamma_1}{dt}, \quad \text{since } \frac{d\nu}{dt} = \cos i_1 \frac{d\gamma_1}{dt},$$

also
$$\frac{d\phi}{dr} = \frac{df}{dr}, \quad \frac{d\phi}{d\vartheta} = \frac{df}{d\theta};$$

therefore from (2) and (3),

$$\frac{d\phi}{d\vartheta}\frac{d\gamma_1}{dt}(1-\cos i_1) + \frac{d\phi_1}{di_1}\frac{di_1}{dt} + \frac{d\phi_1}{d\gamma_1}\frac{d\gamma_1}{dt} = 0,$$

or writing R for ϕ, and $\frac{dR}{ds_1} + \frac{dR}{d\varpi_1}$ for $\frac{dR}{d\vartheta}$,

$$\frac{dR}{di_1}\frac{di_1}{dt} + \left\{\frac{dR}{d\gamma_1} + (1-\cos i_1)\left(\frac{dR}{ds_1} + \frac{dR}{d\varpi_1}\right)\right\}\frac{d\gamma_1}{dt} = 0.$$

St. John's College,
August 21*st*, 1861.

ON THE EQUATIONS OF THE PLANES OF CIRCULAR SECTION OF A CONICOID* REPRESENTED BY TETRAHEDRAL COORDINATES.

By N. M. FERRERS.

TAKE, as the equation of a conicoid,

$$f(\alpha, \beta, \gamma, \delta) \equiv A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + 2\overline{AB}.\alpha\beta + 2\overline{CD}.\gamma\delta + 2\overline{AC}.\alpha\gamma + 2\overline{DB}.\delta\beta + 2\overline{AD}.\alpha\delta + 2\overline{BC}.\beta\gamma = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

and let
$$l\alpha + m\beta + n\gamma + p\delta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

* This term, in place of "surface of the second degree," is adopted from Messrs. Frost and Wolstenholme's recently published "Treatise on Solid Geometry."

be the equation of a plane, passing through the centre of this surface and cutting it in a circle. Our object is to investigate the relations between l, m, n, p, and the coefficients in the equation of the conicoid.

The necessary and sufficient condition that the section should be a circle, is that any three of its radii vectores, measured from the centre, be equal to one another. We will therefore proceed to investigate the conditions for the equality of the radii vectores given by the intersection of the plane (2) with the planes through the centre respectively parallel to

$$\alpha+\beta-\gamma-\delta=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (3),$$

$$\alpha-\beta+\gamma-\delta=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4),$$

$$\alpha-\beta-\gamma+\delta=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

Let $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$, be the coordinates of the centre of (1), and, for shortness, put

$$\alpha-\bar{\alpha}=\xi,\; \beta-\bar{\beta}=\eta,\; \gamma-\bar{\gamma}=\zeta,\; \delta-\bar{\delta}=\theta.$$

Then, if r be the distance from the centre to the point $(\alpha, \beta, \gamma, \delta)$,

$$r^2+ab^2.\xi\eta+cd^2.\zeta\theta+ac^2.\xi\zeta+db^2.\theta\eta+ad^2.\xi\theta+bc^2.\eta\zeta=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

(a, b, c, d being the angular points of the tetrahedron of reference) and

$$f(\xi, \eta, \zeta, \theta)=-f(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) \;\ldots\ldots\ldots\ldots(7).$$

(These two results have been already demonstrated in a paper published in this *Journal*, t. IV., p. 140.)

We have also, identically,

$$\xi+\eta+\zeta+\theta=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8),$$

and, at the intersection of (2) with the plane through the centre, parallel to (3),

$$\xi+\eta-\zeta-\theta=0 \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

$$l\xi+m\eta+n\zeta+p\theta=0 \;\ldots\ldots\ldots\ldots\ldots(10).$$

Hence, combining (8), (9), (10),

$$\frac{\xi}{n-p}=\frac{\eta}{p-n}=\frac{\zeta}{l-m}=\frac{\theta}{m-l}.$$

Hence, by (6) and (7),

$$\frac{r^2}{f(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})}$$

$$=\frac{-ab^2(n-p)^2-cd^2(l-m)^2+(ac^2+db^2+ad^2-bc^2)(n-p)(l-m)}{f(n-p,\,p-n,\,l-m,\,m-l)}$$

$$=\frac{\dfrac{ab^2}{(l-m)^2}+\dfrac{cd^2}{(n-p)^2}-\dfrac{ac^2+db^2-ad^2-bc^2}{(l-m)(n-p)}}{\dfrac{2\overline{AB}-A-B}{(l-m)^2}+\dfrac{2\overline{CD}-C-D}{(n-p)^2}-2\dfrac{\overline{AC}+\overline{DB}-\overline{AD}-\overline{BC}}{(l-m)(n-p)}}.$$

Similar expressions, of course, hold for the values of r corresponding to the intersections of (2) with planes through the centre, parallel to (4) and (5).

By what has been already said, these three values must be equal to each other. Hence we obtain the following relations among the values of l, m, n, p, corresponding to a plane of circular section,

$$\frac{\dfrac{A+B-2\overline{AB}}{(l-m)^2}+\dfrac{C+D-2\overline{CD}}{(n-p)^2}+2\dfrac{\overline{AC}+\overline{DB}-\overline{AD}-\overline{BC}}{(l-m)(n-p)}}{\dfrac{ab^2}{(l-m)^2}+\dfrac{cd^2}{(n-p)^2}-\dfrac{ac^2+db^2-ad^2-bc^2}{(l-m)(n-p)}}$$

$$=\frac{\dfrac{A+C-2\overline{AC}}{(l-n)^2}+\dfrac{D+B-2\overline{DB}}{(p-m)^2}+2\dfrac{\overline{AD}+\overline{BC}-\overline{AB}-\overline{CD}}{(l-n)(p-m)}}{\dfrac{ac^2}{(l-n)^2}+\dfrac{db^2}{(p-m)^2}-\dfrac{ad^2+bc^2-ab^2-cd^2}{(l-n)(p-m)}}$$

$$=\frac{\dfrac{A+D-2\overline{AD}}{(l-p)^2}+\dfrac{B+C-2\overline{BC}}{(m-n)^2}+2\dfrac{\overline{AB}+\overline{CD}-\overline{AC}-\overline{DB}}{(l-p)(m-n)}}{\dfrac{ad^2}{(l-p)^2}+\dfrac{bc^2}{(m-n)^2}-\dfrac{ab^2+cd^2-ac^2-db^2}{(l-p)(m-n)}}.$$

It will be observed that these relations, since they involve only the differences of the coefficients l, m, n, p, hold for *all* circular sections, as well as central ones.

They may be written in a slightly different form, by observing that

$$A+B-2\overline{AB}=\left(\frac{d}{d\alpha}-\frac{d}{d\beta}\right)^2 f,$$

and $\overline{AC}+\overline{DB}-\overline{AD}-\overline{BC}=\left(\frac{d}{d\alpha}-\frac{d}{d\beta}\right)\left(\frac{d}{d\gamma}-\frac{d}{d\delta}\right)f.$

Hence, if $f(\alpha, \beta, \gamma, \delta)=0$, $\phi(\alpha, \beta, \gamma, \delta)=0$ be the equations of two conicoids, and $l\alpha+m\beta+n\gamma+p\delta=0$ that of a plane cutting them in similar and similarly situated curves, we have,

$$\frac{\left(\dfrac{\frac{d}{d\alpha}-\frac{d}{d\beta}}{l-m}+\dfrac{\frac{d}{d\gamma}-\frac{d}{d\delta}}{n-p}\right)^2 f}{\left(\dfrac{\frac{d}{d\alpha}-\frac{d}{d\beta}}{l-m}+\dfrac{\frac{d}{d\gamma}-\frac{d}{d\delta}}{n-p}\right)^2 \phi}=\frac{\left(\dfrac{\frac{d}{d\alpha}-\frac{d}{d\gamma}}{l-n}+\dfrac{\frac{d}{d\delta}-\frac{d}{d\beta}}{p-m}\right)^2 f}{\left(\dfrac{\frac{d}{d\alpha}-\frac{d}{d\gamma}}{l-n}+\dfrac{\frac{d}{d\delta}-\frac{d}{d\beta}}{p-m}\right)^2 \phi}$$

$$=\frac{\left(\dfrac{\frac{d}{d\alpha}-\frac{d}{d\delta}}{l-p}+\dfrac{\frac{d}{d\beta}-\frac{d}{d\gamma}}{m-n}\right)^2 f}{\left(\dfrac{\frac{d}{d\alpha}-\frac{d}{d\delta}}{l-p}+\dfrac{\frac{d}{d\beta}-\frac{d}{d\gamma}}{m-n}\right)^2 \phi}.$$

July 25, 1861.

ON A NEW ANALYTICAL REPRESENTATION OF CURVES IN SPACE.

(Second Paper.)

By A. Cayley.

The employment of a new kind of coordinates for the analytical representation of curves in space is suggested in my former paper under the same title *Journal*, t. III., pp. 225–236 (1859). The idea was as follows: viz. if (x, y, z, w) are current coordinates of a point in space (ordinary point coordinates), and $(\alpha, \beta, \gamma, \delta)$ the coordinates of a particular point, then taking (p, q, r, s, t, u) to represent the minor determinants formed out of the matrix

$$\begin{vmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \end{vmatrix},$$

viz.
$$p=\gamma y-\beta z,\quad s=\delta x-\alpha w,$$
$$q=\alpha z-\gamma x,\quad t=\delta y-\beta w,$$
$$r=\beta x-\alpha y,\quad u=\delta z-\gamma w,$$

values which satisfy identically

$$ps+qt+ru=0,$$

then the equation of a cone passing through a given curve and having for its vertex the arbitrary point $(\alpha, \beta, \gamma, \delta)$, is of the form

$$V=0,$$

V being a homogeneous function of the six new coordinates (p, q, r, s, t, u). And it was proposed to consider $V=0$ as the equation of the curve.

But as remarked in the paper, it is not every function V of the coordinates (p, q, r, s, t, u) which equated to zero, does in fact represent a curve. In order that the equation $V=0$ may represent a curve, it is necessary, that when any infinitesimal variations whatever are given to the constants $(\alpha, \beta, \gamma, \delta)$, thus converting the equation into $V+\delta V=0$, the two equations $V=0$, $\delta V=0$ (considered as equations in ordinary point coordinates) shall represent one and the same curve, whatever the system of infinitesimal variations attributed to $\alpha, \beta, \gamma, \delta$ may be. Let P, Q, R, S, T, U denote the differential coefficients of V in regard to p, q, r, s, t, u respectively, then the equation $\delta V=0$, breaks up into the equations

$$\begin{array}{rrrrl} . & -Ry & +Qz & -Sw & =0, \\ Rx & . & -Pz & -Tw & =0, \\ -Qx & +Py & . & -Uw & =0, \\ Sx & +Ty & +Uw & . & =0, \end{array}$$

and the system composed of these four equations and the equation $V=0$ (considered as equations in ordinary point coordinates) must belong to one and the same curve.

The four equations gave

$$PS+QT+RU=0,$$

a relation between the differential coefficients of V which must be satisfied either identically or in virtue of the equation $V=0$. And this relation existing, any two of the four equations lead to the other two. Attending exclusively to the

coordinates (p, q, r, s, t, u) and considering (x, y, z, w) as mere arbitrary multipliers, the above equation

$$PS + QT + RU = 0$$

is the only relation between the differential coefficients of V which is deducible from the four equations.

But it was noticed that the equation $V = 0$, even when V is a function such that we have (identically or in virtue of the equation $V = 0$) the equation $PS + QT + RU = 0$, does not of necessity represent a curve. Some further relation or relations between the differential coefficients of V must therefore exist, either identically or in virtue of the equation $V = 0$; and such relations can be found by resorting to the second differential $\delta^2 V$ of the function V. In fact not only the equation $\delta V = 0$ but the entire series of relations $\delta^2 V = 0$, $\delta^3 V = 0, \ldots$ should be satisfied by the coordinates of any point of the curve. I find by means of the equation $\delta^2 V = 0$ a plexus of equations, which are consequently necessary, and I am inclined to believe sufficient, in order that the equation $V = 0$ may in fact represent a curve; the equations of the plexus are, it will be seen, very numerous, and certainly only a small number of them are independent, but this is a question which I have not as yet investigated.

Attending to the expressions for p, q, r, s, t, u, we have

$$\begin{aligned}
d_\alpha &= \quad . \quad - y d_r + z d_q - w d_s = (1),\\
d_\beta &= \quad x d_r \quad . \quad - z d_p - w d_t = (2),\\
d_\gamma &= - x d_q + y d_p \quad . \quad - w d_u = (3),\\
d_\delta &= \quad x d_s + y d_t + z d_u \quad . \quad = (4),
\end{aligned}$$

and writing for convenience a, b, c, d instead of $d_\alpha, d_\beta, d_\gamma, d_\delta$, we have

$$d = (1)\,\mathrm{a} + (2)\,\mathrm{b} + (3)\,\mathrm{c} + (4)\,\mathrm{d}.$$

It was in effect by operating on V with this symbol and equating to zero the coefficients of a, b, c, d, that the before-mentioned equations

$$\begin{aligned}
. \quad - Ry + Qz - Sw &= 0,\\
Rx \quad . \quad - Pz - Tw &= 0,\\
- Qx + Py \quad . \quad - Uw &= 0,\\
Sx + Ty + Uz \quad . \quad &= 0,
\end{aligned}$$

were found.

If to these equations we join the equation

$$Ax + By + Cz + Dw = 0,$$

where A, B, C, D are arbitrary multipliers, we can express x, y, z, w in terms of A, B, C, D in such manner as to satisfy the four equations, viz. we have

$$\begin{aligned} x &= \quad . \qquad BU - CT + DP, \\ y &= -AU \quad . \quad + CS + DQ, \\ z &= \quad AT - BS \quad . \quad + DR, \\ w &= -AP - BQ - CR \quad . \ , \end{aligned}$$

and if in the expressions for (1), (2), (3), (4) we substitute for x, y, z, w these values, and form therewith the value of d, which value I will for distinction call $\mathfrak{D}$, we have

$$\mathfrak{D} = \left(\begin{array}{llll} Ud_r + Td_q + Pd_s, & Qd_s - Sd_q \ , & Rd_s - Sd_r \ , & Rd_q - Qd_r \\ Pd_t - Td_p \ , & Ud_r + Qd_t + Sd_p, & Rd_t - Td_r \ , & Pd_r - Rd_p \\ Pd_u - Ud_p \ , & Qd_u - Ud_q \ , & Rd_u + Sd_p + Td_q, & Qd_p - Pd_q \\ Td_u - Ud_t \ , & Ud_s - Sd_u \ , & Sd_t - Td_s \ , & Pd_s + Qd_t + Rd_u \end{array}\right) (A, B, C, D)(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}),$$

viz. $\mathfrak{D}$ is a lineo-linear function of the two sets of indeterminate quantities (A, B, C, D), $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$, the coefficients thereof being the operators

$$Ud_r + Td_q + Pd_s, \quad Qd_s - Sd_q, \ \&c.$$

It may be remarked that we have identically

$$\mathfrak{D}V = (PS + QT + RU)(A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + D\mathrm{d}),$$

since obviously each term such as $(Qd_s - Sd_q)\,V$, which is equal to $QS - SQ$, vanishes identically. The equation $\mathfrak{D}V = 0$ gives therefore only the before-mentioned equation $PS + QT + RU = 0$, which is as it should be.

The equation $\mathfrak{D}^2 V = 0$, is then to be satisfied independently of the values of (A, B, C, D) and $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$, and as $\mathfrak{D}$ contains 16 distinct terms, $\mathfrak{D}^2$ will contain in all $\frac{1}{2}16.17$ or 136 distinct terms. The equation $\mathfrak{D}^2 V = 0$ gives therefore a plexus of 136 equations, and the equations in each succeeding plexus, involved in $\mathfrak{D}^3 V = 0$, $\mathfrak{D}^4 V = 0$ &c. will, of course, be still more numerous.

If $V = 0$ be the plane conic which is the intersection of of the surfaces

$$x^2 + y^2 + z^2 + w^2 = 0,$$

$$ax + by + cz + dw = 0,$$

then we have

$$V = \left(\begin{array}{cccccc} b^2+c^2, & -ab, & -ac, & \cdot\,, & cd\,, & -bd \\ -ba, & c^2+a^2, & -bc, & -cd, & \cdot\,, & ad \\ -ca, & -cb, & a^2+b^2, & bd\,, & -ad, & \cdot \\ \cdot\,, & -cd, & bd\,, & a^2+d^2, & ab\,, & ac \\ cd\,, & \cdot\,, & -ad, & ba\,, & b^2+d^2, & bc \\ -bd, & ad\,, & \cdot\,, & ca\,, & cb\,, & c^2+d^2 \end{array}\right|(p, q, r, s, t, u)^2.$$

The values of P, Q, R, S, T, U (omitting a common factor 2) are

$$P = (b^2+c^2,\ -ab,\ -ac,\ .\ ,\ +cd,\ -bd)\,(p,\ q,\ r,\ s,\ t,\ u),$$

&c.,

and if we proceed to form a term in $\mathfrak{D}^2 V$, say the coefficient of $A^2\mathrm{a}^2$, this is $(Ud_r + Td_q + Pd_s)^2\ V$, or

$$(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(U,\ T,\ P)^2.$$

The coefficient therein of p^2 is

$$(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(-bd,\ cd,\ b^2+c^2)^2,$$

that is, it is

$$\begin{aligned} &(a^2+b^2)\,b^2d^2 && -2cd.\ \ cd\,(b^2+c^2) \\ +&(c^2+a^2)\,c^2d^2 && +2bd.-bd\,(b^2+c^2) \\ +&(a^2+d^2)\,(b^2+c^2)^2 && -2cb.-bd.cd, \end{aligned}$$

where the terms in which (b^2+c^2) does not appear as a factor are together equal to

$$a^2d^2\,(b^2+c^2) + d^2\,(b^2+c^2)^2,$$

the entire expression thus divides by b^2+c^2, the quotient being

$$(a^2+d^2)\,(b^2+c^2) - 2c^2d^2 - 2b^2d^2 + a^2d^2 + d^2\,(b^2+c^2),$$

which is equal to $a^2\,(b^2+c^2+d^2)$, or restoring the factor b^2+c^2, we see that in $\mathfrak{D}^2 V$ the coefficient of $A^2\mathrm{a}^2$ is

$$a^2\,(b^2+c^2+d^2)\,(b^2+c^2)\,p^2 + \text{\&c.}$$

Tho complete value must, it is clear, be of the form

$$a^2\,(b^2+c^2+d^2)\ V + k\,(ps+qt+ru),$$

vanishing in virtue of the equations $V=0$, $ps+qt+ru=0$, and this being so, observing that V contains no term in ps, we have $k =$ coefficient ps in

$$(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)\,(U,\ T,\ P)^2,$$

that is,

$$k = 2\,(a^2+b^2,\ c^2+a^2,\ a^2+d^2,\ -cd,\ +bd,\ -cb)$$
$$(-bd,\ cd,\ b^2+c^2)\,(ca,\ ba,\ 0),$$

or

$$\begin{aligned}\tfrac{1}{2}k = (a^2+b^2).-bd.ca &\ -cd\,\{cd.0+(b^2+c^2)\,ba\}\\ +(c^2+a^2).\quad cd.ba &\ +bd\,\{(b^2+c^2).ca-bd.0\}\\ +(a^2+d^2).\qquad 0 &\ -cb\,\{-bd.ba+cd.ca\},\end{aligned}$$

which is equal to

$$abcd\left\{\begin{array}{ll} -(a^2+b^2) & -(b^2+c^2)\\ +(c^2+a^2) & +(b^2+c^2)\\ & +(b^2-c^2)\end{array}\right\} = 0.$$

The coefficient k consequently vanishes, and therefore in $\mathfrak{D}^2 V$ the coefficient of $A^2\mathfrak{a}^2$ is $a^2(b^2+c^2+d^2)\,V$, but I have not worked out the coefficients of the other terms.

2, Stone Buildings, W.C.
30th October, 1860.

NOTES ON LAMBERT'S ANGLES.

By WILLIAM WALTON, M.A., Trinity College.

LET $ASCA'$ be the major axis of an ellipse, S being a focus, the one nearest to A, and C the centre. From any point M_1 in the major axis draw an ordinate to intersect the ellipse in P_1, and the auxiliary circle in Q_1. For convenience of language, I propose to call Q_1 and M_1 the *circular* and *axial* points, respectively, of the *elliptic* point P_1.

Take an elliptic point P_2 such that the distance of its axial point M_2 from A may be equal to the focal distance of P_1: let Q_2 be the circular point corresponding to P_2. Take any two circular points Q_1', Q_1'', equidistant from Q_1 and on opposite sides of it. Let Q_2', Q_2'', be circular points similarly related to Q_2, and such that the circular arc $Q_2'Q_2Q_2''$ may be equal to the circular arc $Q_1'Q_1Q_1''$. Let P_1', P_1'',

be the elliptic points corresponding to Q_1', Q_1'', and P_2', P_2'', those corresponding to Q_2', Q_2''. Let M_2', M_2'', be the axial points corresponding to Q_2', Q_2''. The relation which the system of points characterized by the suffix (1) bears to the system characterized by the suffix (2) may be repeated in passing from the points characterized by the suffix (2) to points characterized by the suffix (3), and so on for ever.*

I propose to point out certain geometrical properties of the ellipse, which are connected with these correlative systems of points.

Let the angles ACQ_1', ACQ_1'', be denoted by ϕ_1', ϕ_1'', and the angles ACQ_2', ACQ_2'', by ϕ_2', ϕ_2''.

(1). To prove that the sum of the focal distances of two elliptic points P_n', P_n'', is equal to the sum of the distances of the axial points of P_{n+1}', P_{n+1}', from the extremity of the major axis.

Attending to the construction, we see that

$$\begin{aligned}
AM_2' + AM_2'' &= 2a - (CM_2' + CM_2'') \\
&= 2a - a\,(\cos\phi_2' + \cos\phi_2'') \\
&= 2a - 2a\cos\frac{\phi_2'' - \phi_2'}{2}.\cos\frac{\phi_2'' + \phi_2'}{2} \\
&= 2a - 2\,CM_2.\cos\frac{\phi_2'' - \phi_2'}{2} \\
&= 2a - 2\,CM_2.\cos\frac{\phi_1'' - \phi_1'}{2} \\
&= 2a - 2\,(AC - SP_1)\cos\frac{\phi_1'' - \phi_1'}{2} \\
&= 2a - 2ea\cos\frac{\phi_1'' + \phi_1'}{2}\cos\frac{\phi_1'' - \phi_1'}{2} \\
&= 2a - ea\cos\phi_1' - ea\cos\phi_1'' \\
&= SP_1' + SP_1''.
\end{aligned}$$

* In accordance with this construction, the distances of the successive axial points M_1, M_2, M_3 ... from the centre of the ellipse form a geometrical progression of which the eccentricity is the common ratio.

(2). To prove that the distance between two elliptic points P_n', P_n'', is equal to the distance between the axial points of P'_{n+1}, P''_{n+1}.

$$\begin{aligned}(P_1'P_1'')^2 &= a^2 \left\{(\cos\phi_1' - \cos\phi_1'')^2 + (1-e^2)(\sin\phi_1' - \sin\phi_1'')^2\right\}\\ &= a^2\left\{2 - 2\cos(\phi_1' - \phi_1'') - e^2(\sin\phi_1' - \sin\phi_1'')^2\right\}\\ &= 4a^2\left\{\sin^2\frac{\phi_1' - \phi_1''}{2} - e^2\sin^2\frac{\phi_1' - \phi_1''}{2}.\cos^2\frac{\phi_1' + \phi_1''}{2}\right\}\\ &= 4a^2\sin^2\frac{\phi_1' - \phi_1''}{2}.\left(1 - \cos^2\frac{\phi_2' + \phi_2''}{2}\right)\\ &= 4a^2\sin^2\frac{\phi_2'' - \phi_2'}{2}.\sin^2\frac{\phi_2'' + \phi_2'}{2},\end{aligned}$$

$$\begin{aligned}P_1'P_1'' &= 2a\sin\frac{\phi_2'' - \phi_2'}{2}\sin\frac{\phi_2'' + \phi_2'}{2}\\ &= a(\cos\phi_2' - \cos\phi_2'')\\ &= M_2'M_2''.\end{aligned}$$

The investigations which I have pursued in demonstrating these two geometrical properties, point out that the angles ACQ_2', ACQ_2'', are geometrical representations of the subsidiary angles which are involved in Lambert's expression* for the time in which a planet would describe the elliptic arc $P_1'P_1P_1''$ about S regarded as the Sun's place. I am not aware that any geometrical interpretation of the relation of these subsidiary angles to the eccentric angles of the ellipse has been before given.

* These subsidiary angles are not presented explicitly in the result given by Lambert in his treatise entitled *Insigniores orbitæ cometarum proprietates*, p. 125, published in the year 1761: his formula for the time, at which he arrives by a very ingenious method, mostly geometrical, is a constant multiple of the integral

$$\int_{z_{,}}^{z_{,,}} \frac{vdv}{(2av - v^2)^{\frac{1}{2}}},$$

where $z_{,} = \frac{1}{2}(SP_1' + SP_1'' - P_1'P_1'')$, $z_{,,} = \frac{1}{2}(SP_1' + SP_1'' + P_1'P_1'')$.

Lambert performs the integration after expanding the function under the integral sign by the binomial theorem. The angles, which I have called Lambert's angles, and which Laplace in his *Mécanique Céleste*, *liv.* II., chap. 4, has used explicitly, obviously result from mere integration without expansion. The student may see in Hymers' Astronomy the transformation of the expression for the time through the elliptic arc from eccentric angles to Lambert's angles.

(3). To prove that the rectilinear triangles

$$P_1'CP_1'',\ P_2'CP_2'',\ P_3'CP_3'', \ldots$$

are all equal to each other.

$$\begin{aligned}\Delta P_1'CP_1'' &= \tfrac{1}{2}(x_1'y_1'' - x_1''y_1') \\ &= \tfrac{1}{2}ab\,(\sin\phi_1''\cos\phi_1' - \sin\phi_1'\cos\phi_1'') \\ &= \tfrac{1}{2}ab\sin(\phi_1'' - \phi_1') \\ &= \frac{b}{a}.\Delta Q_1'CQ_1'' \\ &= \frac{b}{a}.\Delta Q_2'CQ_2'' \\ &= \Delta P_2'CP_2''.\end{aligned}$$

Hence, by similarity,

$$\Delta P_1'CP_1'' = \Delta P_2'CP_2'' = \Delta P_3'CP_3'' = \ldots.$$

(4). To prove that the distance between the elliptic points P_n', P_n'', is to that between the circular points Q_n', Q_n'', as the diameter conjugate to the diameter through P_n is to the major axis.

By referring to the demonstration of property (2), we see that

$$\begin{aligned}(P_1'P_1'')^2 &= 4a^2\sin^2\frac{\phi_1' - \phi_1''}{2}\left(1 - e^2\cos^2\frac{\phi_1' + \phi_1''}{2}\right) \\ &= (Q_1'Q_1'')^2.\frac{SP_1.HP_1}{(CA)^2} \\ &= \frac{(Q_1'Q_1'')^2.(CD_1)^2}{(CA)^2},\end{aligned}$$

$$P_1'P_1'' = \frac{Q_1'Q_1''.CD_1}{CA},$$

and, similarly, $$P_n'P_n'' = \frac{Q_n'Q_n''.CD_n}{CA}.$$

(5). To prove that, when n becomes infinite, the distance between the elliptic points P_n', P_n'', is equal to that between the circular points Q_n', Q_n''.

Since the chords $Q_1'Q_1''$, $Q_2'Q_2''$, $Q_3'Q_3''$, ... are all evidently equal to each other, let c denote the value of each. Then

$$(P_n'P_n'')^2 = c^2\left(1 - e^2\cos^2\frac{\phi_n' + \phi_n''}{2}\right).$$

Now, by reference to the demonstration of property (2), we see that

$$\cos\frac{\phi_n'+\phi_n''}{2}=e\cos\frac{\phi'_{n-1}+\phi''_{n-1}}{2}=e^2\cos\frac{\phi'_{n-2}+\phi''_{n-2}}{2}=\dots$$
$$=e^{n-1}\cos\frac{\phi_1'+\phi_1''}{2}.$$

Hence we have

$$(P_n'P_n'')^2=c^2\left(1-e^{2n}\cos^2\frac{\phi_1'+\phi_1''}{2}\right).$$

Let $n=\infty$: then, evidently,

$$P_\infty'P_\infty''=c=Q_\infty'Q_\infty''.$$

Combining this result with property (4), we see that $CD_\infty=CA$, and that, consequently, when n is indefinitely increased, Q_n ultimately lies in the axis minor produced.

(6). The chords $P_n'P_n''$, $Q_n'Q_n''$, produced, cut the major axis produced in a single point.

(7). Let T_1, T_2, T_3, ..., be the intersections of the chords $P_1'P_1''$, $P_2'P_2''$, $P_3'P_3''$, ... produced, with the prolongation of the major axis of the ellipse: then will CT_1, CT_2, CT_3, ... form a geometrical progression, the common ratio of which is the reciprocal of the eccentricity of the ellipse.

(8). The intersection of the n^{th} and $(n+1)^{th}$ elliptic chords lies in the line joining the centre of the ellipse with the elliptic point corresponding to a circular point half-way between the n^{th} and $(n+1)^{th}$ circular points.

(9). The rectilinear triangle $P_n'SP_n''$ bears a constant ratio to the rectilinear triangle $Q_n'SQ_n''$.

(10). If u_n represent either the rectilinear triangle $P_n'SP_n''$, the rectilinear triangle $Q_n'SQ_n'$, or the quadrilateral $P_n'P_n''Q_n''Q_n'$, and Δ denote the ordinary symbol of finite differences, then, for all values of n, Δu_{n+1} is a mean proportional between Δu_n and Δu_{n+2}.

I have not supplied the demonstrations of the properties (6), (7), (8), (9), (10), because their truth may so easily be ascertained by the use of the formulæ employed in proving the first five properties. Many other properties, more or less interesting, might be mentioned in connection with Lambert's angles.

April, 1861.

II.

ON THE EXPANSION OF POWERS OF THE TRIGONOMETRICAL RATIOS IN TERMS OF SERIES OF ASCENDING POWERS OF THE VARIABLE.

By Henry M. Jeffery, M.A.

1. The methods, which will be employed in this investigation, depend on various functions of the differences of nothing, and presuppose an acquaintance with the preceding memoir.

Results will be obtained for $(\sin x)^n$ and $(\cos x)^n$, independent of $f(\Delta)\,0^r$, although by its assistance, and agreeing with ordinary expansions. The general terms of the series for $(\operatorname{cosec} x)^n$ and $(\cot x)^n$ involve functions of the form $\left(\frac{D}{\Delta}\right)^n 0^r$: while the expressions for $(\sec x)^n$ and $(\tan x)^n$ require other functions of the form $\frac{1}{(2+\Delta)^n}\,0^r$. It will be further shewn, that each of these functions can be reduced to depend on Bernouilli's numbers, and that, if those of the first form be tabulated, the others can be deduced from them.

2. The general theorem of reduction includes both functions:

$$\frac{(r-1)^{n-1}}{(r+\Delta)^n}\,0^r = \frac{1}{r+\Delta}\,0^r\left\{1+\frac{0}{n-1}\right\}\left\{1+\frac{0}{n-2}\right\}\dots\left\{1+\frac{0}{1}\right\}.$$

By using the fundamental theorem of the preceding memoir, it may easily be found, that

$$\frac{r-1}{(r+\Delta)^n}\,0^r = \frac{1}{(r+\Delta)^{n-1}}\,0^r\left\{1+\frac{0}{n-1}\right\}:$$

For $\frac{1}{(r+\Delta)^{n-1}}\,0^{r+1} = -(n-1)\frac{\frac{d\Delta}{dD}}{(r+\Delta)^n}\,0^r = -(n-1)\frac{1+\Delta}{(r+\Delta)^n}\,0^r$:

and, by pursuing the same course, the formula may be established.

Hence, in the two important cases, when $r=0$, and $r=2$,

$$*\ \frac{D^n}{\Delta^n}\,0^s=(-1)^{n-1}\,s^{(n)}\,\frac{1}{\Delta}\,0^{s-n}\left(1+\frac{0}{1}\right)\left(1+\frac{0}{2}\right)\dots\left(1+\frac{0}{n-1}\right),$$

$$\frac{1}{(2+\Delta)^n}\,0^s=\frac{1}{2+\Delta}\,0^s\left(1+\frac{0}{1}\right)\left(1+\frac{0}{2}\right)\dots\left(1+\frac{0}{n-1}\right).$$

By aid of the table for values of $\zeta^r 0^s$ given in the preceding memoir, we may express $\frac{D^n}{\Delta^n}\,0^s$ in a form adapted for computation.

$$\frac{D^n}{\Delta^n}\,0^s=\frac{s^{(n)}}{\lfloor\underline{n-1}}\,\frac{D}{\Delta}\left\{\frac{0^{s-n+1}}{s-n+1}\cdot\frac{\zeta}{\lfloor\underline{1}}-\frac{0^{s-n+2}}{s-n+2}\cdot\frac{\zeta^2}{\lfloor\underline{2}}+\dots\pm\frac{0^s}{s}\cdot\frac{\zeta^n}{\lfloor\underline{n}}\right\}0^n.$$

If $n>s$, a preferable form may be obtained:

$$\frac{1}{\lfloor\underline{n-1}}\cdot\frac{D}{\Delta}\left\{\frac{\zeta 0^n}{\lfloor\underline{1}}.D^{n-1}+(n-s-1)\,\frac{\zeta^2 0^n}{\lfloor\underline{2}}.D^{n-2}\right.$$
$$\left.+(n-s-1)(n-s-2)\,\frac{\zeta^3 0^n}{\lfloor\underline{3}}\,D^{n-3}+\dots\right\}0^s.$$

* Dr. Brinkley, who first introduced the numbers of the form $\Delta^m 0^n$ as *data* in analysis, has developed the function $\left(\frac{t}{\epsilon^t-1}\right)^n$ in powers of t in the *Phil. Trans.*, 1807, 1.

Several months after this memoir of mine was placed in the Editor's hands, I have seen for the first time and studied Mr. Blissard's treatise on Generic Equations in the last number of this *Journal*, and find he has investigated with great success the properties of the functions $\left(\frac{2}{2+\Delta}\right)^m 0^n$ and $\left(\frac{D}{\Delta}\right)^m 0^n$, which that writer has denoted U_n and V_n. One of his two methods for reducing these functions, viz. that marked (XI.) and (XII.), is identical with the theorem in the text.

A third method has since been suggested by Prof. Sylvester's process for expressing a negative power of a series in terms of positive powers of the same series. (*Math. Journal*, Dec. 1855.)

$$\frac{r^n}{(r+\Delta)^n}\,0^s=-\frac{n}{1}\,\frac{\lfloor\underline{n+s}}{\lfloor\underline{n+1}.\lfloor\underline{s-1}}\left(1+\frac{\Delta}{r}\right)0^s$$
$$+\frac{n(n+1)}{1.2}\cdot\frac{\lfloor\underline{n+s}}{\lfloor\underline{n+2}.\lfloor\underline{s-2}}\left(1+\frac{\Delta}{r}\right)^2 0^s-\dots.$$

It may be here observed, that the usual notation for Bernouilli's numbers is objectionable, from its being unmeaning: $\frac{D}{\Delta}0^{2m}$ should always be written instead of $(-1)^{m-1}B_{2m-1}$.

3. Another expression, but not so convenient for computing, may be obtained for $\frac{D^n}{\Delta^n}0^s$.

Since $x^s = \left\{x^{(1)}\frac{\Delta}{\lfloor 1} + x^{(2)}\frac{\Delta^2}{\lfloor 2} + \ldots + x^{(s)}\frac{\Delta^s}{\lfloor s}\right\}0^s$,

$$\frac{D^n}{\Delta^n}0^s = \frac{D^n}{\Delta^n}\left\{0^{(1)}\frac{\Delta}{\lfloor 1} + 0^{(2)}\frac{\Delta^2}{\lfloor 2} + \ldots + 0^{(s)}\frac{\Delta^s}{\lfloor s}\right\}0^s,$$

$$= D^n\left\{\frac{0^{(n+1)}}{\lfloor n+1}\Delta + \frac{0^{(n+2)}}{\lfloor n+2}\Delta^2 + \ldots + \frac{0^{(n+s)}}{\lfloor n+s}\Delta^s\right\}0^s,$$

by theorem (D),

$$= \zeta^n\left\{\frac{0^{n+1}}{\lfloor n+1}\Delta + \frac{0^{n+2}}{\lfloor n+2}\Delta^2 + \ldots + \frac{0^{n+s}}{\lfloor n+s}\Delta^s\right\}0^s,$$

by Art. 6, Mem. 1.

4. The annexed table was calculated both from the formula contained in the preceding article 2: and also by successive steps from the expression

$$\frac{D^n}{\Delta^n}0^s = -\frac{D^{n-1}}{\Delta^{n-1}}\left\{s.0^{s-1} + \frac{s-n+1}{n-1}.0^s\right\},$$

which may be obtained, as before, from the general theorem (B).

It is noteworthy that for the particular value $n-1$,

$$\frac{D^n}{\Delta^n}0^{n-1} = (-1)^{n-1}.\lfloor n-1.$$

Another simple test of accuracy may be derived from the formula:

$$\left(\frac{D}{\Delta}\right)^n.(2.0+n)^{2m+1} = 0.$$

The proof of this proposition will be contained in Art. 12.

	$\frac{D}{\Delta}$	$\frac{D^2}{\Delta^2}$	$\frac{D^3}{\Delta^3}$	$\frac{D^4}{\Delta^4}$	$\frac{D^5}{\Delta^5}$	$\frac{D^6}{\Delta^6}$	$\frac{D^7}{\Delta^7}$	$\frac{D^8}{\Delta^8}$	$\frac{D^9}{\Delta^9}$	$\frac{D^{10}}{\Delta^{10}}$
0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3	$-\frac{7}{2}$	-4	$-\frac{9}{2}$	-5
0^2	$\frac{1}{6}$	$\frac{5}{6}$	2	$\frac{11}{3}$	$\frac{35}{6}$	$\frac{17}{2}$	$\frac{35}{3}$	$\frac{46}{3}$	$\frac{39}{2}$	$\frac{145}{6}$
0^3	0	$-\frac{1}{2}$	$-\frac{9}{4}$	-6	$-\frac{25}{2}$	$-\frac{45}{2}$	$-\frac{147}{4}$	-56	-81	$-\frac{225}{2}$
0^4	$-\frac{1}{30}$	$\frac{1}{10}$	$\frac{19}{10}$	$\frac{251}{30}$	24	$\frac{274}{5}$	$\frac{1624}{15}$	$\frac{967}{5}$	$\frac{3207}{10}$	$\frac{3013}{6}$
0^5	0	$\frac{1}{6}$	$-\frac{3}{4}$	-9	$-\frac{475}{12}$	-120	-294	$-\frac{1876}{3}$	$-\frac{2403}{2}$	$-\frac{4275}{2}$
0^6	$\frac{1}{42}$	$-\frac{5}{42}$	$-\frac{16}{21}$	$\frac{221}{42}$	$\frac{4315}{84}$	$\frac{19087}{84}$	720	$\frac{13068}{7}$	$\frac{29531}{7}$	$\frac{180920}{21}$
0^7	0	$-\frac{1}{6}$	$\frac{5}{4}$	$\frac{11}{3}$	$-\frac{475}{12}$	$-\frac{1375}{4}$	$-\frac{36799}{24}$	-5040	-13698	-32575
0^8	$-\frac{1}{30}$	$\frac{7}{30}$	$\frac{19}{30}$	$-\frac{199}{18}$	$-\frac{329}{18}$	$\frac{9829}{30}$	$\frac{237671}{90}$	$\frac{1070017}{90}$	40320	114064
0^9	0	$\frac{3}{10}$	$-\frac{63}{20}$	$\frac{3}{5}$	$\frac{395}{4}$	$\frac{171}{2}$	$-\frac{59829}{20}$	$-\frac{114562}{5}$	$-\frac{2082753}{20}$	-362880
0^{10}	$\frac{5}{66}$	$-\frac{15}{22}$	$-\frac{3}{11}$	$\frac{707}{22}$	$-\frac{2385}{44}$	$-\frac{41065}{44}$	$-\frac{15365}{66}$	$\frac{330157}{11}$	$\frac{9751299}{44}$	$\frac{134211265}{132}$

Additional importance is attached to the symbol $\frac{D^n}{\Delta^n}0^s$, since the successive summation of series by approximation depends on this form.

$$\Sigma^n u_x = \frac{1}{D^n}\cdot\frac{D^n}{\Delta^n}u_x = \frac{1}{D^n}\cdot\frac{D^n}{\Delta^n}.\varepsilon^{0D}u_x$$

$$= \frac{D^n}{\Delta^n}\left\{\left(\frac{d}{dx}\right)^{-n} + \frac{0}{\underline{|1}}\cdot\left(\frac{d}{dx}\right)^{-n+1} + \ldots + \frac{0^n}{\underline{|n}} + \frac{0^{n+1}}{\underline{|n+1}}\cdot\frac{d}{dx} + \ldots\right\} u_x.$$

5. It may be proved, that, as in the particular case of Bernouilli's numbers, so universally, $\frac{D^n}{\Delta^n}0^s$ is ultimately divergent; but $\frac{1}{\underline{|s}}\cdot\frac{D^n}{\Delta^n}0^s$ is ultimately convergent, when the odd and even terms are considered separately.

By Art. 2, the limit of $\frac{1}{\lfloor 2s+2} \cdot \frac{D^n}{\Delta^n} 0^{2s+2} \div \frac{1}{\lfloor 2s} \cdot \frac{D^n}{\Delta^n} 0^{2s}$

$$= \frac{(2s+2)^{(n)}}{\lfloor 2s+2} \cdot \frac{D}{\Delta} \left\{ \frac{0^{2s+2}}{2s+2} + \frac{0^{2s}}{2s} \cdot \frac{\zeta^{n-2}}{\lfloor n-2} 0^n + \dots \right\}$$

$$\div \frac{(2s)^{(n)}}{\lfloor 2s} \cdot \frac{D}{\Delta} \left\{ \frac{0^{2s}}{2s} + \dots \right\}$$

$$= -\frac{(2s+2)(2s+1)}{(2s-n+2)(2s-n+1)} \cdot \frac{2}{(2s+2)(2\pi)^{2s+2}} \div \frac{2}{2s.(2\pi)^{2s}} = -\frac{1}{(2\pi)^2}$$

ultimately, if $2s$ be taken at least $> n\pi$, as a limitation necessary, if the terms involving $\frac{\zeta^{n-2}}{\lfloor n-2} 0^n$, $\frac{\zeta^{n-4}}{\lfloor n-4} . 0^n$, ... are to be neglected, since these functions of n increase by four dimensions for several terms of the series.

A similar proof will establish the existence of convergency in the limit of $\frac{1}{\lfloor 2s+3} \frac{D^n}{\Delta^n} 0^{2s+3} \div \frac{1}{\lfloor 2s+1} \cdot \frac{D^n}{\Delta^n} . 0^{2s+1}$.

The ratio of the consecutive terms is ultimately $\frac{n.(n-1)}{4s}$, and $\frac{s}{n.(n-1).\pi^2}$, indefinitely small and indefinitely great.

It is easy to deduce the fact of divergency, when the factorials in the coefficients are withdrawn.

6. To find a limit for the remainder of the series for $\Sigma^2 u_x$ after $(2n+1)$ terms.

The series, neglecting the subscript x, is

$$\iint u\,dx^2 - \int u\,dx + \frac{1}{\lfloor 2} \cdot \frac{D^2}{\Delta^2} 0^2 . u + \frac{1}{\lfloor 3} \cdot \frac{D^2}{\Delta^2} 0^3 \frac{du}{dx} + \dots$$

$$+ \frac{1}{\lfloor 2n} \frac{D^2}{\Delta^2} 0^{2n} \frac{d^{2n-2}u}{dx^{2n-2}} + R.$$

The alternate terms of the remainder may be conveniently grouped together,

$$R = \frac{1}{\lfloor 2n+1} \frac{D^2}{\Delta^2} 0^{2n+1} \frac{d^{2n-1}u}{dx^{2n-1}} + \frac{1}{\lfloor 2n+3} \frac{D^2}{\Delta^2} 0^{2n+3} \frac{d^{2n+1}u}{dx^{2n+1}} + \dots$$

$$+ \frac{1}{\lfloor 2n+2} \frac{D^2}{\Delta^2} 0^{2n+2} \frac{d^{2n}u}{dx^{2n}} + \frac{1}{\lfloor 2n+4} \frac{D^2}{\Delta^2} 0^{2n+4} \frac{d^{2n+2}u}{dx^{2n+2}} + \dots;$$

therefore

$$-R=\frac{1}{\lfloor\underline{2n}}\,\frac{D}{\Delta}\,0^{2n}\,\frac{d^{2n-1}u}{dx^{2n-1}}+\frac{1}{\lfloor\underline{2n+2}}\,\frac{D}{\Delta}\,0^{2n+2}\,\frac{d^{2n+1}u}{dx^{2n+1}}+\dots$$

$$+\frac{2n+1}{\lfloor\underline{2n+2}}\,\frac{D}{\Delta}\,0^{2n+2}\,\frac{d^{2n}u}{dx^{2n}}+\frac{2n+3}{\lfloor\underline{2n+4}}\cdot\frac{D}{\Delta}\,0^{2n+4}\,\frac{d^{2n+2}u}{dx^{2n+2}}+\dots.$$

$\Big\{$For, by Art. 2, $\dfrac{D^2}{\Delta^2}\,0^{2n}=-(2n-1)\,\dfrac{D}{\Delta}\,0^{2n}$

$$:\frac{D^2}{\Delta^2}\,0^{2n+1}=-(2n+1)\,\frac{D}{\Delta}\,0^{2n}.\Big\}$$

The summations of these portions depend on the solutions of differential equations of these respective types:

$$\frac{d^2U}{dx^2}+(2m\pi)^2\,U=(-1)^{n-1}\,\frac{2}{(2m\pi)^{2n-2}}\cdot\frac{d^{2n-1}u}{dx^{2n-2}}.$$

$$\frac{d^4V}{dx^4}+2\,(2m\pi)^2\,\frac{d^2V}{dx^2}+(2m\pi)^4\,V$$

$$=(-1)^n\,2\left\{\frac{2n+1}{(2m\pi)^{2n-2}}\cdot\frac{d^{2n}u}{dx^{2n}}+\frac{2n-1}{(2m\pi)^{2n}}\,\frac{d^{2n+2}u}{dx^{2n+2}}\right\}.$$

By proceeding in the usual way, the numerical limits of the sums of all the forms of U and V are found to be

$$\frac{1}{2\pi}\cdot\frac{1}{\lfloor\underline{2n-2}}\,\frac{D}{\Delta}\,0^{2n-2}\cdot\frac{d^{2n-2}u}{dx^{2n-2}},$$

$$\frac{2n+1}{2\lfloor\underline{2n}}\,\frac{D}{\Delta}\,0^{2n}\,\frac{d^{2n-2}u}{dx^{2n-2}}-\frac{2n-1}{2\lfloor\underline{2n+2}}\cdot\frac{D}{\Delta}\,0^{2n+2}\,\frac{d^{2n}u}{dx^{2n}}$$

$$+\frac{1}{4\pi}\,\frac{2n+1}{\lfloor\underline{2n}}\cdot\frac{D}{\Delta}\,0^{2n}\cdot\frac{d^{2n-1}u}{dx^{2n-1}}-\frac{1}{4\pi}\,\frac{2n-1}{\lfloor\underline{2n+2}}\,\frac{D}{\Delta}\,0^{2n+2}\cdot\frac{d^{2n+1}u}{dx^{2n+1}}.$$

Remark. The determination of limits of the remainder, after taking certain terms of $\Sigma^n u_x$, will require the solutions of the equation $\left(\dfrac{d^2}{dx^2}+a^2\right)^r u=X$, for all values of r from 1 to n.

7. Another expression for $\Sigma^n u_x$ may be easily proved by taking the successive differences:

$$\lfloor\underline{n}\,.\,\Sigma^{n+1}u_x=x^{(n)}\Sigma u_{x-n}-\frac{n}{1}.x^{(n-1)}\Sigma\,(x-n+1)^{(1)}u_{x-n}$$

$$+\frac{n\,(n-1)}{1.2}\,x^{(n-2)}\Sigma\,(x-n+2)^{(2)}\,u_{x-n}+\dots.$$

Ex. To find an approximate value for $1^1.2^2\ldots x^x$, when x is very large.

Write $\log x$ for u_x in this identity:

$$\Sigma x u_x = (x-1)\,\Sigma u_x - \Sigma^2 u_x.$$

$$\log(1^1.2^2\ldots x^x) = c + x\log x$$

$$+\,(x-1)\left\{\left(x-\frac{1}{2}\right)\log x - x + \frac{1}{12x} - \frac{1}{360x^3} + \ldots\right\}$$

$$-\left\{\left(\frac{x^2}{2} - x + \frac{5}{12}\right)\log x - \frac{3x^2}{4} + x - \frac{1}{12x} - \frac{1}{240x^2} + \frac{1}{360x^3} - \ldots\right\}$$

$$= C + \left(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{12}\right)\log x - \frac{x^2}{4}$$

$$-\frac{\lfloor 1}{\lfloor 4}\,\frac{D}{\Delta}\,0^4.\frac{1}{x^2} - \frac{\lfloor 3}{\lfloor 6}\,\frac{D}{\Delta}\,0^6.\frac{1}{x^4} - \frac{\lfloor 5}{\lfloor 8}\,\frac{D}{\Delta}\,0^8.\frac{1}{x^6} - \ldots,$$

where on trial C is found to be $\cdot 24875$.

Hence $$1^1.2^2\ldots x^x = \varepsilon^{\cdot 24875 - \frac{x^2}{4}}\,x^{\frac{x^2}{2}+\frac{x}{2}+\frac{1}{12}},$$

A second constant does not appear in the result, since the substitution of 1 for x causes $(x-1)\,\Sigma u_x$ to vanish.

The value of the constant may be also thus obtained. Since

$$1^1.2^2\ldots x^x = \varepsilon^{C-\frac{x^2}{4}}.x^{\frac{x^2}{2}+\frac{x}{2}+\frac{1}{12}},$$

if x be indefinitely great,

$$\frac{4^4.8^8\ldots(4x)^{4x}}{2^2.6^6\ldots(4x-2)^{4x-2}} = \frac{\varepsilon^{4C-x^2}\,(4x)^{2x^2+2x}.x^{\frac{1}{3}}}{\varepsilon^{-2C-x^2}\,(4x)^{2x^2}\,2^{\frac{1}{6}}x^{-\frac{1}{6}}} = \varepsilon^{6C}\,(4x)^{2x}\,x^{\frac{1}{2}}.2^{-\frac{1}{6}}.$$

Now the evaluation of the expression

$$\log\frac{1^2.3^2.4^4.4^4.5^6.7^6.8^8.8^8\ldots \textit{ad infinitum}}{2^2.2^2.3^4.5^4.6^6.6^6.7^8.9^8\ldots\ldots\ldots\ldots\ldots\ldots}$$

depends by Cauchy's theorem for the test of convergency on the definite integral (given in Boole's *Finite Differences*, p. 67)

$$\int_0^\infty dx\left\{(4x+2)\log\frac{(4x+1)(4x+3)}{(4x+2)^2} - (4x+4)\log\frac{(4x+3)(4x+5)}{(4x+4)^2}\right\},$$

or $$\frac{(4x+1)(4x+3)}{8}\log(4x+1) - (2x+1)^2\log(4x+2)$$

$$-\frac{4x+3}{2}\log(4x+3)$$

$$+\,(2x+2)^2\log(4x+4) - \frac{(4x+3)(4x+5)}{8}\log(4x+5).$$

The integral vanishes at the upper limit: by varying the lower limit, the required value of the ratio of the infinite products is found to be .660.

But this ratio may also be written:

$$\frac{4^4.4^4.8^8.8^8\ldots(4x)^{4x}.(4x)^{4x}}{2^2.2^2.6^6.6^6\ldots(4x-2)^{4x-2}.(4x-2)^{4x-2}}\times\frac{1}{(4x+1)^{4x}}\times\frac{1^2.5^2.9^2\ldots(4x-3)^2}{3^2.7^2.11^2\ldots(4x-1)^2},$$

and $$\frac{3.7.11\ldots(4x-1)}{1.5.9\ldots(4x-3)}\cdot\frac{1}{\sqrt{(x)}}=\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)},$$

since the limit of $\dfrac{n(n+1)\ldots(n+r-1)}{1.2\ldots r}$ is $\dfrac{r^{n-1}}{\Gamma(n)}$, (Todhunter's *Integral Calculus*, p. 220).

By making the necessary substitutions,

$$\varepsilon^6 C.(4x)^{2x}.x^{\frac{1}{4}}.2^{-\frac{1}{6}}=(4x+1)^{2x}.x^{\frac{1}{4}}.\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\times\sqrt{(.660)};$$

therefore $\varepsilon^6 C=2^{\frac{1}{6}}.\varepsilon^{\frac{1}{4}}.\dfrac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\times\sqrt{(.660)}$. Hence $C=.2487$.

8. The values of $\dfrac{1}{(2+\Delta)^n}0^r$ can be derived from those of $\dfrac{D^n}{\Delta^n}0^r$ in the following manner.

Since $$\frac{1}{\varepsilon^D+1}=\frac{1}{\varepsilon^D-1}-\frac{2}{\varepsilon^{2D}-1},$$

it may be proved that

$$\frac{1}{(\varepsilon^D-1)^n}+(-1)^n.\frac{1}{(\varepsilon^D+1)^n}=\left(\frac{2}{\varepsilon^{2D}-1}\right)^n+n.\frac{1}{\varepsilon^{2D}-1}\left(\frac{2}{\varepsilon^{2D}-1}\right)^{n-2}$$
$$+\frac{n(n-3)}{1.2}\left(\frac{1}{\varepsilon^{2D}-1}\right)^2\left(\frac{2}{\varepsilon^{2D}-1}\right)^{n-4}$$
$$+\frac{n.(n-4)(n-5)}{1.2.3}\left(\frac{1}{\varepsilon^{2D}-1}\right)\left(\frac{2}{\varepsilon^{2D}-1}\right)^{n-6}+\ldots.$$

The last term is $2.\left(\dfrac{1}{\varepsilon^{2D}-1}\right)^{\frac{1}{2}n}$, if n is even:

and $n.\left(\dfrac{1}{\varepsilon^{2D}-1}\right)^{\frac{1}{2}(n-1)}.\left(\dfrac{2}{\varepsilon^{2D}-1}\right)$, if n is odd.

It may be noted, that these coefficients are numerically the same as those in the expansion of $2\cos n\theta$ in terms of powers of $2\cos\theta$.

Hence, by observing that

$$\left(\frac{2D}{\varepsilon^{2D}-1}\right)^s f(0) = \left(\frac{D}{\varepsilon^{D}-1}\right)^s \varepsilon^{0.2D} f(0) = \frac{D^s}{\Delta^s} f(20),$$

we obtain the connecting formula:

$$(-1)^n . \frac{D^n}{(2+\Delta)^n} 0^r = -\frac{D^n}{\Delta^n} 0^r + \left(\frac{2D}{\varepsilon^{2D}-1}\right)^n 0^r + \frac{n}{2} . \left(\frac{2D}{\varepsilon^{2D}-1}\right)^{n-1} D.0^r$$

$$+ \frac{n.(n-3)}{1.2.2^2} . \left(\frac{2D}{\varepsilon^{2D}-1}\right)^{n-2} D^2.0^r + \dots.$$

Or,

$$(-1)^n . r^{(n)} \frac{1}{(2+\Delta)^n} 0^{r-n} = (2^r - 1) \frac{D^n}{\Delta^n} 0^r + nr.2^{r-2} . \frac{D^{n-1}}{\Delta^{n-1}} 0^{r-1}$$

$$+ \frac{n.(n-3)}{1.2} . r.(r-1)\, 2^{r-4} \frac{D^{n-2}}{\Delta^{n-2}} 0^{r-2} + \dots.$$

Cor. 1. Hence it may be shewn that the denominator of $\frac{D}{\Delta} 0^{2s}$ or of B_{2s-1} measures $2(2^{2s}-1)$.

1. $$\frac{D}{\Delta} 0^{2s} = \left(-\frac{\Delta}{2} + \frac{\Delta^2}{3} - \frac{\Delta^3}{4} + \dots\right) 0^{2s}:$$

now $\frac{\Delta^n}{\lfloor n} 0^{2s}$ is always integral, and $\frac{\Delta^3}{\lfloor 3} 0^{2s}$ or its equivalent $\frac{3^{2s-1} - 2.2^{2s-1} + 1}{2}$ is an even number (*Quarterly Journal*, p. 370): hence the denominator contains only the first power of 2.

2. The denominators of $\frac{1}{2+\Delta} 0^{2s-1}$ consist exclusively of powers of 2: and $-\frac{2^{2s}-1}{2s} . \frac{D}{\Delta} 0^{2s}$ is its equivalent. Hence the denominators of Bernouilli's numbers must measure $2^{2s}-1$.

Cor. 2. The following table of values was calculated by the theorem of reduction given in Art. 2:

$$\frac{1}{(2+\Delta)^n} 0^s = \frac{1}{(2+\Delta)^{n-1}} \left\{0^s + \frac{0^{s+1}}{n-1}\right\}.$$

To test its accuracy, a formula may be used, which can be derived from Art. 13,

$$\left(\frac{2}{2+\Delta}\right)^n \sin(2.0+n)\, x = 0, \text{ or } \frac{2^n}{(2+\Delta)^n} (2.0+n)^{2m+1} = 0.$$

Another test may be supplied by Art. 9, 2.

	$\frac{1}{2+\Delta}$	$\frac{1}{(2+\Delta)^2}$	$\frac{1}{(2+\Delta)^3}$	$\frac{1}{(2+\Delta)^4}$	$\frac{1}{(2+\Delta)^5}$	$\frac{1}{(2+\Delta)^6}$	$\frac{1}{(2+\Delta)^7}$	$\frac{1}{(2+\Delta)^8}$	$\frac{1}{(2+\Delta)^9}$	$\frac{1}{(2+\Delta)^{10}}$
0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{16}$	$-\frac{1}{8}$	$-\frac{5}{64}$	$-\frac{3}{64}$	$-\frac{7}{256}$	$-\frac{1}{64}$	$-\frac{9}{1024}$	$-\frac{5}{1024}$
0^2	0	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{5}{32}$	$\frac{15}{128}$	$\frac{21}{256}$	$\frac{7}{128}$	$\frac{9}{256}$	$\frac{45}{2048}$
0^3	$\frac{1}{8}$	$\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{25}{128}$	$-\frac{27}{128}$	$-\frac{49}{256}$	$-\frac{5}{32}$	$-\frac{243}{2048}$	$-\frac{175}{2048}$
0^4	0	$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{9}{32}$	$-\frac{5}{64}$	$\frac{15}{128}$	$\frac{63}{256}$	$\frac{77}{256}$	$\frac{153}{512}$	$\frac{135}{512}$
0^5	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{32}$	$\frac{13}{16}$	$\frac{125}{128}$	$\frac{99}{128}$	$\frac{49}{128}$	$-\frac{1}{64}$	$-\frac{81}{256}$	$-\frac{125}{256}$
0^6	0	$\frac{17}{16}$	$\frac{51}{32}$	$\frac{21}{32}$	$-\frac{65}{64}$	$-\frac{75}{32}$	$-\frac{357}{128}$	$-\frac{77}{32}$	$-\frac{99}{64}$	$-\frac{2385}{4096}$
0^7	$\frac{17}{16}$	$\frac{17}{16}$	$-\frac{45}{16}$	$-\frac{107}{16}$	$-\frac{425}{64}$	$-\frac{171}{64}$	$\frac{343}{128}$	$\frac{55}{8}$	$\frac{35559}{4096}$	$\frac{33475}{4096}$
0^8	0	$-\frac{31}{4}$	$-\frac{93}{8}$	$\frac{3}{16}$	$\frac{635}{32}$	$\frac{2055}{64}$	$\frac{3759}{128}$	$\frac{7399}{512}$	$-\frac{4689}{1024}$	$-\frac{41585}{2048}$
0^9	$-\frac{1}{4}$	$-\frac{31}{4}$	$\frac{567}{16}$	$\frac{629}{8}$	$\frac{3025}{64}$	$-\frac{1053}{64}$	$-\frac{53459}{512}$	$-\frac{19487}{128}$	$-\frac{289413}{2048}$	$-\frac{173425}{2048}$
0^{10}	0	$\frac{691}{8}$	$\frac{2073}{16}$	$-\frac{1107}{16}$	$-\frac{12445}{32}$	$-\frac{135105}{256}$	$-\frac{171423}{512}$	$\frac{23379}{256}$	$\frac{260973}{512}$	$\frac{3046815}{4096}$

9. The sums of any number of terms of the functions $\frac{D^n}{\Delta^n}\, 0^r$, $\frac{1}{(2+\Delta)^n}\, 0^r$, are connected by the following theorems.

By Art. 3, Cor. (Mem. 1), it appears that

$$\left\{\iint \frac{D^n}{\Delta^n}\, d\Delta^2\right\} 0^{r+2} = (1+\Delta)^2 \frac{D^n}{\Delta^n}\,(0^r + 0^{r-1} + \ldots + 0 + 1).$$

1. Hence by Art. 7, Cor. (Mem. 1), p. 366,

$$\frac{D^n}{\Delta^n}\,\Sigma 0^r + 2.\frac{D^{n-1}}{\Delta^{n-1}}\,\Sigma r 0^{r-1} + \frac{D^{n-2}}{\Delta^{n-2}}\,\Sigma r(r-1)\,0^{r-2}$$

$$= \left\{\iint \frac{\zeta^n}{D^n}\,\varepsilon^{0\Delta} d\Delta^2\right\} 0^{r+2} = \frac{\zeta^n}{D^n}\left\{\frac{\Delta^2}{\lfloor 2} + 0.\frac{\Delta^3}{\lfloor 3} + \ldots + 0^r\,\frac{\Delta^{r+2}}{\lfloor r+2}\right\} 0^{r+2}$$

$${}^*= \frac{\zeta^n}{\lfloor n}\left\{0^n\,\frac{\Delta^2}{\lfloor 2} + \frac{0^{n+1}}{n+1}\cdot\frac{\Delta^3}{2.3} + \ldots + \frac{0^{n+r}}{(n+1)\ldots(n+r)}\cdot\frac{\Delta^{r+2}}{(r+1)(r+2)}\right\} 0^{r+2}.$$

Ex. When $n=1$, we can express the difference of Bernouilli's numbers:

$$\frac{D}{\Delta}\,\Sigma 0^r + 1 + \frac{r(r+1)}{1.2} = \left\{\frac{\Delta^2}{1^2.2} - \frac{\Delta^3}{2^2.3} + \ldots \pm \frac{\Delta^{r+2}}{(r+1)^2(r+2)}\right\} 0^{r+2}.$$

* In my last MS. I had employed the Hebrew *Daleth* (ד) and not ζ to represent $\log(1+D)$: this explanation may render intelligible the analogy described in Art. 4 (Mem. 1) p. 365.

Otherwise $\left\{\iint \frac{D}{\Delta}\, d\Delta^2\right\} 0^{2n+2} = \left\{(1+\Delta) \int \frac{D}{\Delta}\, d\Delta\right\} 0^{2n+1}$

by the fundamental theorem (B),

$$= \frac{D}{\Delta}\, 0^{2n} + \left\{\Delta \int \frac{D}{\Delta}\, d\Delta\right\} 0^{2n+1}.$$

But $\qquad \left\{\iint \frac{D}{\Delta}\, d\Delta^2\right\} 0^{2n+2} = \frac{D}{\Delta}\, \Sigma 0^{2n} + 1 + \frac{2n\,(2n+1)}{1.2}.$

Hence $B_1 - B_2 + \ldots + (-1)^n B_{2n-2} = \left\{\Delta \int \frac{D}{\Delta}\, d\Delta\right\} 0^{2n+1} - (2n^2 + n + \frac{3}{2})$

$$= \left(\frac{\Delta^2}{1^2} - \frac{\Delta^3}{2^2} + \ldots\right) 0^{2n+1} - (2n^2 + n + \tfrac{3}{2}).$$

2. By proceeding similarly, it may be seen that

$$\frac{1}{(n-1)(n-2)} \cdot \frac{1}{(2+\Delta)^{n-2}} \cdot 0^{r+2} = \left\{\iint \frac{d\Delta^2}{(2+\Delta)^n}\right\} 0^{r+2}$$

$$= \frac{(1+\Delta)^2}{(2+\Delta)^n} \{0^r + 0^{r-1} + \ldots + 0 + 1\} - \frac{1}{(n-1)\, 2^{n-1}}.$$

Hence $\qquad \frac{1}{(n-1)\, 2^{n-1}} + \frac{1}{(n-1)(n-2)} \cdot \frac{1}{(2+\Delta)^{n-2}}\, 0^{r+2}$

$$= \left\{\frac{1}{(2+\Delta)^{n-2}} - \frac{2}{(2+\Delta)^{n-1}} + \frac{1}{(2+\Delta)^n}\right\} \Sigma 0^r.$$

10. I now proceed to apply the preceding results, to determine the expansions of powers of the trigonometrical ratios, and shall commence with the simplest cases.

To find the general term of $(\sin x)^n$.

$$\left(\frac{\sin x}{x}\right)^n = \left\{\frac{\varepsilon^{2x\sqrt{(-1)}} - 1}{2x\sqrt{(-1)}}\right\}^n . \varepsilon^{-nx\sqrt{(-1)}} = \frac{\Delta^n}{D^n}\, \varepsilon^{-\frac{1}{2}nD} . \varepsilon^{0.2x\sqrt{(-1)}}$$

(since by Herschel's theorem $f(x)\ xf(D)\ \varepsilon^{0.x}$)

$$= \frac{\Delta^n}{D^n}\, \varepsilon^{(2.0-n)x\sqrt{(-1)}} = \frac{\Delta^n}{D^n} \cos(20 - n)\, x,$$

since the imaginary portion must vanish.

The $(m+1)^{\text{th}}$ term of $(\sin x)^n = (-1)^m . \frac{\Delta^n}{D^n}\, (2.0 - n)^{2m} . \frac{x^{2m+n}}{|\underline{2m}}$,

and by integration, $= (-1)^m . \frac{1}{2^n} . \frac{x^{2m+n}}{\lfloor 2m+n} (\varepsilon^D - 1)^n (2.0 - n)^{2m+n}$

$$= (-1)^m . \frac{1}{2^n} . \frac{x^{2m+n}}{\lfloor 2m+n} \left\{ n^{2m+n} - n(n-2)^{2m+n} + \frac{n.(n-1)}{1.2} .(n-4)^{2m+n} - \ldots + n^{2m+n} \right\}.$$

$$= (-1)^m \frac{1}{2^{n-1}} . \frac{x^{2m+n}}{\lfloor 2m+n} \left\{ n^{2m+n} - n(n-2)^{2m+n} + \ldots \text{ to } \frac{n}{2} \text{ or } \frac{n+1}{2} \text{ terms} \right\}.$$

11. To find the general term of $(\cos x)^n$.

By proceeding as before, it will be found that

$$(2 \cos x)^n = (1 + \varepsilon^D)^n \cos(2.0 - n)\, x \text{ or } (2 + \Delta)^n \cos(20 - n)\, x.$$

The $(m+1)^{\text{th}}$ term of this expansion is

$$(-1)^m . (1 + \varepsilon^D)^n . (2.0 - n)^{2m} . \frac{x^{2m}}{\lfloor 2m},$$

or $(-1)^m\, 2 . \frac{x^{2m}}{\lfloor 2m} \left\{ n^{2m} + n.(n-2)^{2m} + \frac{n.(n-1)}{1.2} (n-4)^{2m} + \ldots \text{ to } \frac{n}{2} \text{ or } \frac{n+1}{2} \text{ terms} \right\}.$

12. To find the general term of $(\operatorname{cosec} x)^n$,

$$(x \operatorname{cosec} x)^n = \left\{ \frac{2x \sqrt{(-1)}}{\varepsilon^{2x\sqrt{(-1)}} - 1} \right\}^n . \varepsilon^{nx\sqrt{(-1)}} = \frac{D^n}{\Delta^n} \varepsilon^{\frac{1}{2}nD} . \varepsilon^{0.2x\sqrt{(-1)}}$$

$$= \frac{D^n}{\Delta^n} \cos(2.0 + n)\, x : \text{ and } \frac{D^n}{\Delta^n} \sin(20 + n)\, x = 0.$$

The values of the coefficients of the various powers of x can be obtained from the table of integral values of $\frac{D^n}{\Delta^n} 0^r$: if n be fractional, the functions $\left(\frac{D}{\Delta}\right)^n$ and $\left(\frac{\Delta}{D}\right)^n$ should be expanded by the usual algebraical processes in ascending powers of Δ or D.

When $n = 1$, it can be seen that

$$\pi x \operatorname{cosec} \pi x = 1 + H_1 x^2 + H_2 x^4 + \ldots,$$

where $H_1, H_2, \ldots$ denote the homogeneous products of different dimensions, which can be formed of the series $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$.

For $\dfrac{\pi x}{\sin \pi x} = \left\{1 - \left(\dfrac{x}{1}\right)^2\right\}^{-1} . \left\{1 - \left(\dfrac{x}{2}\right)^2\right\}^{-1} . \left\{1 - \left(\dfrac{x}{3}\right)^2\right\}^{-1} \ldots.$

13. $$\left(\frac{\sec x}{2}\right)^n = \frac{1}{(2+\Delta)^n} \cos(2.0+n)\,x,$$

which may be otherwise written by Art. 2,

$$\frac{1}{2+\Delta}\left\{1+\frac{0}{1}\right\}\left\{1+\frac{0}{2}\right\}\ldots\left\{1+\frac{0}{n-1}\right\}\cos(2.0+n)\,x.$$

In Art. 8 it has been shewn, that $\dfrac{1}{(2+\Delta)^n}$ can be expanded in a descending series of powers of $\dfrac{D}{\Delta}$: and its values are calculated for integer values of n as far as ten.

Ex.

$$\left(\frac{\sec x}{2}\right)^2 = \frac{1}{2+\Delta}\left\{(1+0)-(1+0)^3.\frac{(2x)^2}{\lfloor 2} + (1+0)^5.\frac{(2x)^4}{\lfloor 4} - \ldots\right\}$$

$$= \frac{D}{\Delta}\left\{-\left(0+\frac{30^2}{2}\right) + \left(0+\frac{90^2}{2}+\frac{15}{4}0^4\right)2x^2+\ldots\right\}$$

$$= \frac{1}{4}+\frac{x^2}{4}+\frac{x^4}{6}+\frac{17x^6}{180}+\ldots,$$

since $$\frac{1}{2+\Delta}f(0) = \frac{1}{\Delta}\{f(0)-2f(2.0)\}.$$

14. Another more convenient expression exists for $\sec x$:

$$x\sec x = -\frac{D}{\Delta}\sin(4.0+1)\,x.$$

For $$x\sec x = \frac{2x\varepsilon^{x\sqrt{(-1)}}}{\varepsilon^{2x\sqrt{(-1)}}-1} - \frac{4x\varepsilon^{x\sqrt{(-1)}}}{\varepsilon^{4x\sqrt{(-1)}}-1}$$

$$= \frac{1}{\sqrt{(-1)}}.\left\{\frac{D\varepsilon^{\frac{1}{2}D}}{\varepsilon^D-1}.\varepsilon^{0.2x\sqrt{(-1)}} - \frac{D\varepsilon^{\frac{1}{4}D}}{\varepsilon^D-1}.\varepsilon^{0.4x\sqrt{(-1)}}\right\}$$

$$= \frac{D}{\Delta}\{\sin(2.0+1)\,x - \sin(4.0+1)\,x\}.$$

In this form, the general term coincides with that given in Herschel's *Examples on Finite Differences*, p. 95, but is capable of being further simplified.

For $\dfrac{D}{\Delta}\sin(2.0+1)\,x = 0$, by Art. 12.

15. Euler has shewn that

$$\sec x = \frac{2^2}{\pi} C_1 + \frac{2^4}{\pi^3} C_3 x^2 + \frac{2^6}{\pi^5} C_5 x^4 + \ldots,$$

where $$C_{2n+1} = \frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \ldots.$$

The theorem may be thus simply established:

$$\tan\frac{\pi}{4}(1+x) = \frac{1+x}{1-x} \cdot \frac{1-\dfrac{x}{3}}{1+\dfrac{x}{3}} \cdot \frac{1+\dfrac{x}{5}}{1-\dfrac{x}{5}} \ldots.$$

For all the solutions of $\sin\frac{\pi}{4}(1+x) = 0$ are comprised in the formula $x = 4n - 1$, and of $\cos\frac{\pi}{4}(1+x) = 0$ in the formula $x = 4n + 1$.

$$\log\tan\frac{\pi}{4}(1+x) = 2\left(\frac{C_1 x}{1} + \frac{C_3 x^3}{3} + \frac{C_5 x^5}{5} + \ldots\right).$$

Differentiating, $\frac{\pi}{4}\sec\frac{\pi x}{2} = C_1 + C_3 x^2 + C_5 x^4 + \ldots.$

Hence $$c_1 = \frac{\pi}{4} : \quad c_3 = \frac{\pi^3}{32} : \quad c_5 = \frac{5\pi^5}{1536} : \ldots.$$

Again, since $\cos\frac{\pi x}{2} = \left\{1 - \left(\frac{x}{1}\right)^2\right\}\left\{1 - \left(\frac{x}{3}\right)^2\right\}\left\{1 - \left(\frac{x}{5}\right)^2\right\}\ldots,$

we may obtain another expression for $\sec\frac{\pi x}{2}$,

$$1 + H_1 x^2 + H_2 x^4 + \ldots,$$

where H_1, H_2 denote the homologous products of different dimensions of the infinite series $\frac{1}{1^2}, \frac{1}{3^2}, \frac{1}{5^2}, \ldots.$

16. The ratio of the two infinite products

$$\frac{2^3.4^3.5^5.5^5.6^7.8^7\ldots}{2.3^3.3^3.4^5.6^5.7^7.7^7\ldots} = \frac{1}{C_1}\,\varepsilon^{-\frac{C_2}{C_1}},$$

where C_1, C_2 have the meaning assigned them in the last article.

$$\frac{\pi x}{2}\sec\frac{\pi x}{2} = 2C_1 x + 2C_3 x^3 + \ldots$$

$$= \frac{\frac{2x}{1^2}}{1-\left(\frac{x}{1}\right)^2} - 3.\frac{\frac{2x}{3^2}}{1-\left(\frac{x}{3}\right)^2} + \ldots;$$

therefore $\log\dfrac{\left\{1-\left(\frac{x}{3}\right)^2\right\}^3\left\{1-\left(\frac{x}{7}\right)^2\right\}^7\ldots}{\left\{1-\left(\frac{x}{1}\right)^2\right\}\left\{1-\left(\frac{x}{5}\right)^2\right\}^5\ldots} = \int_0^x dx\,\frac{\pi x}{2}\sec\frac{\pi x}{2}$

$$= x\log\tan\frac{\pi}{4}(1+x) - \int_0^x dx\log\tan\frac{\pi}{4}(1+x).$$

By equating x to unity,

$$\log\frac{2^3.4^3.5^5.5^5.6^7.8^7\ldots}{2.3^3.3^3.4^5.6^5.7^7.7^7\ldots} = \log\left[\frac{1-x}{\left\{\cot\frac{\pi}{4}(1+x)\right\}^x}\right]_{x=1} - \int_0^1 dx\log\tan\frac{\pi}{4}(1+x)$$

$$= \log\frac{4}{\pi} + \frac{4}{\pi}\int_0^{\frac{1}{4}\pi} dx\log\tan x.$$

By integrating by parts,

$$= \log\frac{4}{\pi} - \frac{4}{\pi}\int_0^{\frac{1}{4}\pi}\frac{d.\tan x}{\tan x}\left(\tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \ldots\right)$$

$$= \log\frac{1}{C_1} - \frac{1}{C_1}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right).$$

Hence we obtain the required evaluation,

$$\frac{1}{C_1}\varepsilon^{-\frac{C_2}{C_1}} = \frac{4}{\pi}\varepsilon^{-1.16\ldots} = .3966.$$

No expression for C_{2n} is known: but C_2 may be found in terms of $c_1, c_3, \ldots$:

$$\frac{C_2}{2C_1} = \frac{2}{\pi}C_2 = -\frac{2}{\pi}\int_0^{\frac{1}{4}\pi} dx\log\tan x = \frac{1}{2}\int_0^1 d\theta\log\tan\pi(1+\theta)$$

$$= \frac{C_1}{1.2} + \frac{C_3}{3.4} + \frac{C_5}{5.6} + \ldots = \log 2 + \frac{1-C_1}{1.2} + \frac{1-C_3}{3.4} + \frac{1-C_5}{5.6} + \ldots,$$

which series is very convergent.

17. $\dfrac{\cos\alpha}{1^2+\kappa^2} - \dfrac{3\cos 3\alpha}{3^2+\kappa^2} + \dfrac{5\cos 5\alpha}{5^2+\kappa^2} - \ldots = \dfrac{\pi}{4}\cdot\dfrac{\varepsilon^{\kappa\alpha}+\varepsilon^{-\kappa\alpha}}{\varepsilon^{\frac{1}{2}\kappa\pi}+\varepsilon^{-\frac{1}{2}\kappa\pi}},$

$$\frac{\kappa \sin\alpha}{1^2+\kappa^2} - \frac{\kappa \sin 3\alpha}{3^2+\kappa^2} + \frac{\kappa \sin 5\alpha}{5^2+\kappa^2} - \ldots = \frac{\pi}{4} \cdot \frac{\varepsilon^{\kappa\alpha} - \varepsilon^{-\kappa\alpha}}{\varepsilon^{\frac{1}{2}\kappa\pi} + \varepsilon^{-\frac{1}{2}\kappa\pi}}.$$

For convenience, call these series $\frac{u+v}{2}$, $\frac{u-v}{2}$; then it will be found that

$$\frac{du}{d\alpha} - \kappa u = -\sin\alpha + \sin 3\alpha - \sin 5\alpha + \ldots = 0,$$

$$\frac{dv}{d\alpha} + \kappa v = 0.$$

It may be remarked, that on the simplicity of expression for the determining series of sines or cosines depends the summation of series such as those proposed in pairs.

Integrating the differential equations,

$$u = C\varepsilon^{\kappa\alpha}: \quad v = C'\varepsilon^{-\kappa\alpha}.$$

Write $\alpha = 0$ in the proposed series:

$$\frac{C+C'}{2} = \frac{1}{1^2+\kappa^2} - \frac{3}{3^2+\kappa^2} + \frac{5}{5^2+\kappa^2} - \ldots = \frac{\pi}{2} \cdot \frac{1}{\varepsilon^{\frac{1}{2}\kappa\pi} + \varepsilon^{-\frac{1}{2}\kappa\pi}},$$

since $\frac{\pi}{4}\sec\frac{\pi x}{2} = \frac{1}{1^2-x^2} - \frac{3}{3^2-x^2} + \frac{5}{5^2-x^2} - \ldots$, by Art. 15.

$$C - C' = 0;$$

therefore $$C = C' = \frac{\pi}{4} \cdot \frac{1}{\varepsilon^{\frac{1}{2}\kappa\pi} + \varepsilon^{-\frac{1}{2}\kappa\pi}}.$$

18. To expand $(\cot x)^n$ in a series of ascending powers of x. Since

$$x^n \{\cot x - \sqrt{(-1)}\}^n = \left\{\frac{2x\sqrt{(-1)}}{\varepsilon^{2x\sqrt{(-1)}} - 1}\right\}^n = \frac{D^n}{\Delta^n}\, \varepsilon^{0.2x\sqrt{(-1)}},$$

by equating real and imaginary values on both sides,

$$x^n . \frac{\cos nx}{\sin^n x} \text{ or } x^n \left\{\cot^n x - \frac{n(n-1)}{1.2}\cot^{n-2} x + \ldots\right\} = \frac{D^n}{\Delta^n}\cos 2.0.x,$$

$$x^n . \frac{\sin nx}{\sin^n x} \text{ or } x^n \left\{n \cot^{n-1} x - \frac{n(n-1)(n-2)}{1.2.3}\cot^{n-3} x + \ldots\right\}$$

$$= -\frac{D^n}{\Delta^n}\sin 2.0.x.$$

It may be easily seen that $(x \cot x)^n$ can be expanded in a series of the form

$$\left\{c_0 \left(\frac{D}{\Delta}\right)^n + n(n-1)\, c_2 x^2 \left(\frac{D}{\Delta}\right)^{n-2} + n(n-1)(n-2)(n-3)\, c_4 x^4 \left(\frac{D}{\Delta}\right)^{n-4} + \ldots\right\} \cos 2.0.x,$$

where the coefficients $c_0, c_2, c_4, \ldots$ are constant, and can be determined by a definite law.

Apply the previous theorem to the case of $(x \cot x)^{n+2}$, and make the requisite substitutions: then $(x \cot x)^{n+2} =$

$$\left\{\begin{array}{l} c_0 \left(\frac{D}{\Delta}\right)^{n+2} + (n+2)(n+1)\,\frac{c_0}{\lfloor 2}\, x^2.\left(\frac{D}{\Delta}\right)^n \\ +(n+2)\ldots(n-1)\frac{c_2}{\lfloor 2}x^4\left(\frac{D}{\Delta}\right)^{n-2} + (n+2)\ldots(n-3)\frac{c_4}{\lfloor 2}x^6\left(\frac{D}{\Delta}\right)^{n-4} + \ldots \\ -(n+2)\ldots(n-1)\frac{c_0}{\lfloor 4}x^4\left(\frac{D}{\Delta}\right)^{n-2} - (n+2)\ldots(n-3)\frac{c_2}{\lfloor 4}x^6\left(\frac{D}{\Delta}\right)^{n-4} - \ldots \\ \qquad\qquad + (n+2)\ldots(n-3)\frac{c_0}{\lfloor 6}x^6\left(\frac{D}{\Delta}\right)^{n-4} + \ldots \end{array}\right\} \cos 2.0x.$$

By equating the coefficients in this expansion with those in the equivalent series

$$\left\{c_0 \left(\frac{D}{\Delta}\right)^{n+2} + (n+2)(n+1)\, c_2 x^2 \left(\frac{D}{\Delta}\right)^n + (n+2)\ldots(n-1)\, c_4 x^4 \left(\frac{D}{\Delta}\right)^{n-2} + \ldots\right\} \cos 2.0.x,$$

the law of formation is recognized.

$$c_2 = \frac{c_0}{\lfloor 2}: \quad c_4 = \frac{c_2}{\lfloor 2} - \frac{c_0}{\lfloor 4}: \quad c_6 = \frac{c_4}{\lfloor 2} - \frac{c_2}{\lfloor 4} + \frac{c_0}{\lfloor 6}.$$

These connecting equations are comprised in the following identity:

$$(c_0 + c_2 x^2 + c_4 x^4 + \ldots)\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\right) = 1,$$

or $$c_0 + c_2 x^2 + c_4 x^4 + \ldots = \sec x.$$

But $x \sec x$ has been shewn to be equal to

$$-\frac{D}{\Delta} \sin(4.0+1)\, x;$$

hence $-c_0 = \frac{D}{\Delta}(4.0+1): \quad c_2 = \frac{D}{\Delta}\,\frac{(4.0+1)^3}{\lfloor 3} : \ldots : \ldots.$

The following explicit expression is thus obtained:

$$-(x\cot x)^n = \frac{D}{\Delta}(4.0+1)\left(\frac{D}{\Delta}\right)^n \cos 2.0.x$$

$$\mp n(n-1)\frac{D}{\Delta}\frac{(4.0+1)^3}{\lfloor 3}\left(\frac{D}{\Delta}\right)^{n-2} x^2\cos 2.0.x + \ldots.$$

From the other series of powers of $\cot x$, we may similarly obtain a second explicit value:

$$-nx^n\cot^{n-1}x = \left\{d_0\left(\frac{D}{\Delta}\right)^n + n.(n-1)\,d_2x^2\left(\frac{D}{\Delta}\right)^{n-2}\right.$$

$$\left. + n(n-1)(n-2)(n-3)\,d_4x^4\left(\frac{D}{\Delta}\right)^{n-4} + \ldots\right\}\sin 2.0.x,$$

where d_0, d_2, d_4, ... are the coefficients of $x\operatorname{cosec} x$, or

$$x\frac{D}{\Delta}\cos(20+1)\,x.$$

Ex.
$$(x\cot x)^3 = \frac{D^3}{\Delta^3}\left\{1 - \frac{2^2x^2}{\lfloor 2}.0^2 + \ldots\right\}$$

$$+ 3\frac{D}{\Delta}\left\{x^2 - \frac{2^2x^4}{\lfloor 2}.0^2 + \ldots\right\}$$

$$= 1 - x^2 + \frac{4}{15}x^4 + \frac{1}{945}x^6 + \ldots.$$

19. In like manner, *mutatis mutandis*, we obtain two corresponding expressions for $(\tan x)^n$,

$$x^n\{\tan x + \sqrt{(-1)}\}^n = \left\{\frac{2x\sqrt{(-1)}}{\varepsilon^{2x\sqrt{(-1)}}+1}\right\}^n = \frac{D^n}{(2+\Delta)^n}\,\varepsilon^{0.2x\sqrt{(-1)}},$$

$$-(x\tan x)^n = \left\{\frac{D}{\Delta}(4.0+1)\left(\frac{D}{2+\Delta}\right)^n\right.$$

$$\left. - n(n-1)\frac{D}{\Delta}\frac{(4.0+1)^3}{\lfloor 3}.\left(\frac{D}{\Delta}\right)^{n-2}x^2 + \ldots\right\}\cos 2.0.x,$$

$$-nx^n\tan^{n-1}x = \left\{\left(\frac{D}{2+\Delta}\right)^n\right.$$

$$\left. - n(n-1)\frac{D}{\Delta}\frac{(2.0+1)^2}{\lfloor 2}.\left(\frac{D}{2+\Delta}\right)^{n-2}x^2 + \ldots\right\}\sin 2.0.x.$$

Cheltenham,
Dec. 1860.

ON THE INTERNAL PRESSURES WITHIN AN ELASTIC SOLID.

By JAMES W. WARREN, A.B.

THE principal results in the theory of Elastic Solids, although known more or less completely to Cauchy and chiefly developed by him, yet have been extended and simplified much since they came from the hands of that great analyst. I purpose in the following communication to consider the elasticity of solids from a more physical point of view, than, as far as I am aware, writers have as yet regarded them.

Mr. Rankine in the former series of this *Journal*, and subsequently Mr. J. B. Phear of Clare College, have given demonstrations, the former geometrical, the latter analytical, of the existence of three principal axes of elasticity in every solid of constant homogenity. I intend to give new demonstrations of this and some other results, and also to add a few results I believe new.

I commence with Cauchy's theorem of the tetrahedron of elasticity quoted by Mr. Phear, viz.,

$$\left.\begin{aligned} F.\cos\alpha' &= N_1.\cos\alpha + T_3.\cos\beta + T_2.\cos\gamma \\ F.\cos\beta' &= T_3.\cos\alpha + N_2.\cos\beta + T_1.\cos\gamma \\ F.\cos\gamma' &= T_2.\cos\alpha + T_1.\cos\beta + N_3.\cos\gamma \end{aligned}\right\}\text{.....(1)},$$

where F is a resultant force exercised on the largest face of the tetrahedron. N_1, N_2, &c. denote pressures normal and transverse, as used by Mr. Rankine and others, and F_1 is resultant of N_1, T_2, and T_3; F_2 of N_2, T_3, and T_1; F_3 of N_3, T_1, and T_2. Write $F.\cos\alpha' = x$, $F.\cos\beta' = y$, $F.\cos\gamma' = z$, and let x', y', and z' denote components of F resolved along F_1, F_2, and F_3; therefore by the theory of prejections we get immediately

$$\left.\begin{aligned} x &= \frac{N_1}{F_1}x' + \frac{T_3}{F_2}y' + \frac{T_2}{F_3}z' \\ y &= \frac{T_3}{F_1}x' + \frac{N_2}{F_2}y' + \frac{T_1}{F_3}z' \\ z &= \frac{T_2}{F_1}x' + \frac{T_1}{F_2}y' + \frac{N_3}{F_3}z' \end{aligned}\right\}\text{...............(2)},$$

therefore by comparing (1) and (2) we get

$$\cos\alpha = \frac{x'}{F_1};\quad \cos\beta = \frac{y'}{F_2};\quad \cos\gamma = \frac{z'}{F_3}.$$

Let x'', y'', z'' be current coordinates of the largest face of the tetrahedron, α'', β'', γ'' the angles its axis makes with F_1, F_2, and F_3; therefore its equation referred to these may be written

$$x''.\cos\alpha''+y''.\cos\beta''+z''.\cos\gamma''=\text{constant}.$$

Let P denote this axis, therefore

$$\begin{aligned}\cos\alpha''&=\cos(\widehat{P.\text{axis of } x', \textit{i.e. } F_1})=\cos(\widehat{P;\ N_1}).\cos(\widehat{F_1;\ N_1})\\&\quad+\cos(\widehat{P;\ N_2}).\cos(\widehat{F_1;\ N_2})+\cos(\widehat{P_1;\ N_3})\cos(\widehat{F_1;\ N_3})\\&=\frac{N_1x'}{F_1^2}+\frac{T_3y'}{F_1.F_2}+\frac{T_2z'}{F_3.F_1}=\frac{x}{F_1};\end{aligned}$$

similarly for $\cos\beta''$ and $\cos\gamma''$, therefore the equation of this plane is

$$\begin{aligned}&\left(\frac{N_1}{F_1}x'+\frac{T_3}{F_2}y'+\frac{T_2}{F_3}z'\right).\frac{x''}{F_1}\\+&\left(\frac{T_3}{F_1}x'+\frac{N_2}{F_2}y'+\frac{T_1}{F_2}z'\right).\frac{y''}{F_2}\\+&\left(\frac{T_2}{F_1}x'+\frac{T_1}{F_2}y'+\frac{N_3}{F_3}z'\right).\frac{z''}{F_3}=\text{constant},\end{aligned}$$

but this is the plane conjugate to point $x'y'z'$ in the surface

$$\frac{N_1}{F_1^2}x^2+\frac{N_2}{F_2^2}y^2+\frac{N_3}{F_3^2}.z^2+\frac{2T_1}{F_2F_3}yz+\frac{2T_2}{F_1F_3}xz+\frac{2T_3}{F_1F_2}xy=\text{constant}\;\ldots\ldots\ldots\ldots(3),$$

and therefore the resultant elastic force on any element plane at a point of solid has the diameter of this ellipsoid conjugate to the plane for its direction, but every surface of the second order has three mutually orthogonal planes whose conjugate diameters are respectively perpendicular to them, therefore *at every point of a solid there are three mutually orthogonal planes whose resultant elastic forces are respectively perpendicular to each plane.* Now take these lines for axes of coordinates, and remembering that our surface has a physical significance, and that now $N_1'=F_1'$; $N_2'=F_2'$; $N_3'=F_3'$; $T_1'=T_2'=T_3'=0$; it becomes, by writing Φ_1 for F_1', &c.,

$$\frac{x^2}{\Phi_1}+\frac{y^2}{\Phi_2}+\frac{z^2}{\Phi_3}=\text{constant}\;\ldots\ldots\ldots\ldots\ldots(4),$$

which, according to the signs of Φ_1, Φ_2, and Φ_3, is an ellipsoid or hyperboloid.

System (1) now becomes

$$F.\cos\alpha' = \Phi_1.\cos\alpha,$$

$$F.\cos\beta' = \Phi_2.\cos\beta,$$

$$F.\cos\gamma' = \Phi_3.\cos\gamma;$$

but
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

therefore
$$\frac{1}{F^2} = \frac{\cos^2\alpha'}{\Phi_1^2} + \frac{\cos^2\beta'}{\Phi_2^2} + \frac{\cos^2\gamma'}{\Phi_3^2} = \frac{1}{p^2.r^2};$$

therefore, we see that, *the elastic force exercised on a unit area of any diametral plane of this ellipsoid, is given in direction by its conjugate diameter, and in magnitude by* $\dfrac{\textit{length of conjugate diameter}}{\textit{area of diametral plane}}$.

Hence immediately follows the theorem of "Reciprocity," since it is a known result, easily proved, and first I believe by Mr. Mac Cullagh, (*On the Dynamical Theory of Crystalline Reflection and Refraction*, Trans. R. I. A., 1846, p. 24), that if r and p denote a radius vector and perpendicular on its tangent plane for an ellipsoid r' and p', two more for the same surface, and

$$\omega \text{ angle between } r \text{ and } p',$$

$$\omega' \text{ angle between } r' \text{ and } p;$$

then
$$rp.\cos\omega = r'.p'.\cos\omega';$$

therefore
$$F.\cos\omega = F'.\cos\omega';$$

or the projection of F on the normal to F''s element plane equals the projection of F' on the normal to F's element plane.

Imagine our ellipsoid to become very small, so as that any diametral plane has sensibly elastic force constant at every point, then *the elastic forces exercised on any two diametral planes are as their respective conjugate diameters.*

So far my chief object has been to deduce the fundamental properties of an elastic solid by the aid of *one* geometrical representative alone; the making use of two, as M. Lamé has done, gives unnecessary obscurity to this beautiful theory. I shall then, in future, designate this surface by the name "ellipsoid (or hyperboloid) of elasticity," and omit as useless the consideration of the surface so designated by M. Lamé. Imagine any plane through our

"ellipsoid of elasticity," the elastic force for the unit of the area of this plane is equal to $p \times r$, and the normal elastic force on this plane

$$= F \times \frac{p}{r} = p^2;$$

therefore, we get the theorem, *The normal force for any diametral plane of the ellipsoid is as the square of corresponding perpendiculars*, but the squares on any three mutually orthogonal perpendiculars when added is constant, and, therefore, $N_1 + N_2 + N_3$ for any three mutually orthogonal planes is constant, the radius vector of the ellipsoid reciprocal to the ellipsoid of elasticity is equal to p, and therefore we get

$$N_m = \cos^2\alpha . \Phi_1 + \cos^2\beta . \Phi_2 + \cos^2\gamma . \Phi_3 .$$

Again,

$$T_1 = F_2 . \cos\widehat{(F_2N_1)} = F_3 . \cos\widehat{(F_3N_1)} = R_2 . P_2 . \cos\widehat{(R_2P_1)}$$
$$= R_3 . P_3 . \cos\widehat{(R_3P_1)},$$

where P_1, P_2, and P_3; R_1, R_2, and R_3 are perpendiculars and radii vectores, corresponding to any three mutually orthogonal directions of N_1, N_2, N_3; or, *Drawing three lines mutually orthogonal through the centre of the "ellipsoid of elasticity," the tangential force corresponding to one of these directions, is equal to one of the perpendiculars corresponding to either of the other directions multiplied by the perpendicular let fall from the extremity of its radius vector on a plane of which the third line is the axis, i.e.*

$$T_m = P_3 . \left(\frac{P_2 x_2}{a^2} . x_3 + \frac{P_2 y_2}{b^2} . y_3 + \frac{P_2 z_2}{c^2} . z_3 \right) = a^2 . \cos\alpha . \cos\alpha'$$
$$+ b^2 . \cos\beta . \cos\beta' + c^2 . \cos\gamma . \cos\gamma',$$

or, in terms of the ellipsoid of elasticity,

$$T_m = \cos\alpha_2 . \cos\alpha_3 . \Phi_1 + \cos\beta_2 . \cos\beta_3 . \Phi_2 + \cos\gamma_2 . \cos\gamma_3 . \Phi_3 .$$

Let us now change the coordinate axes from the directions Φ_1, Φ_2, Φ_3 to N_1, N_2, N_3, we have to substitute

$$x = \cos\alpha_1 . x + \cos\alpha_2 . y + \cos\alpha_3 . z,$$
$$y = \cos\beta_1 . x + \cos\beta_2 . y + \cos\beta_3 . z,$$
$$z = \cos\gamma_1 . x + \cos\gamma_2 . y + \cos\gamma_3 . z;$$

substituting these in the equation of the ellipsoid reciprocal to the ellipoid of elasticity, viz.,

$$\Phi_1 . x^2 + \Phi_2 . y^2 + \Phi_3 . z^2 = \text{constant},$$

we obtain

$$x^2.(\Phi_1.\cos^2\alpha_1+\Phi_2.\cos^2\beta_1+\Phi_3.\cos^2\gamma_1)$$
$$+2yz.(\Phi_1.\cos\alpha_2.\cos\alpha_3+\Phi_2.\cos\beta_2.\cos\beta_3+\Phi_3.\cos\gamma_2.\cos\gamma_3)+\&c.$$

equals a constant, or from formulae,

$$N_1.x^2+N_2.y^2+N_3.z^2$$
$$+2T_1.yz+2T_2.xz+2T_3.xy=\text{constant}\ldots\ldots\ldots\ldots(5),$$

and, therefore, the transformed equation of the ellipsoid of elasticity (which is the reciprocal of the above surface) is

$$(N_2N_3-T_1^2).x^2+(N_3N_1-T_2^2)\,y^2+(N_1N_2-T_3^2).z^2$$
$$+2\,(T_2T_3-N_1T_1).yz+2\,(T_3T_1-N_2T_2)\,xz$$
$$+2\,(T_1T_2-N_3T_3)\,xy=\text{constant}\ldots\ldots\ldots\ldots(6).$$

We have thus got the equation of the ellipsoid of elasticity in the three forms (3), (4), and (6); we know that in an ellipsoid the sum of the squares of the reciprocals of any three mutually orthogonal radii equal a constant, therefore

$$N_1N_2+N_2N_3+N_3N_1-T_1^2-T_2^2-T_3^2,$$

is a "physical invariant," as well as $N_1+N_2+N_3$; and the discriminant of (5), or,

$$N_1N_2N_3+2\,T_1T_2T_3-N_1T_1^2-N_2T_2^2-N_3T_3^2,$$

and now by the same geometrical means that we obtained N_1, &c. from

$$\Phi_1.x^2+\Phi_2.y^2+\Phi_3.z^2=\text{constant},$$

we may get N_1', &c. from

$$N_1x^2+N_2y^2+N_3z^2+2\,T_1yz+2\,T_2xz+2\,T_3xy=\text{constant},$$

or by remembering that N_1' is inversely proportional to the square of the radius drawn in its direction; result by either method is

$$N_m'=N_1.\cos^2\alpha_1+N_2.\cos^2\beta_1+N_3.\cos^2\gamma_1$$
$$+2\,T_1.\cos\beta_1.\cos\gamma_1+2\,T_2.\cos\alpha_1.\cos\gamma_1+2\,T_3.\cos\alpha_1.\cos\beta_1,$$
$$T_m'=N_1.\cos\alpha_2.\cos\alpha_3+N_2.\cos\beta_2.\cos\beta_3+N_3.\cos\gamma_2.\cos\gamma_3$$
$$+\,T_1.(\cos\beta_2.\cos\gamma_3+\cos\gamma_2.\cos\beta_3)$$
$$+\,T_2.(\cos\beta_1.\cos\gamma_3+\cos\gamma_1.\cos\beta_3)$$
$$+\,T_3.(\cos\beta_1.\cos\gamma_2+\cos\beta_2.\cos\gamma_1).$$

And similarly we get from our reciprocal ellipsoid, *i.e.* ellipsoid of elasticity,

$$
\begin{aligned}
(N_1'.N_2' - T_1'^2) = (N_2N_3 - T_1^2).\cos^2\alpha_1& \\
+ (N_3N_1 - T_2^2).\cos^2\beta_1 &+ (N_1N_2 - T_3^2).\cos^2\gamma_1 \\
+ 2\,(T_2T_3 - N_1T_1).\cos\beta_1.\cos\gamma_1& \\
+ 2\,(T_3T_1 - N_2T_2).\cos\alpha_1.\cos\gamma_1& \\
+ 2\,(T_1T_2 - N_3T_3).\cos\alpha_1.\cos\beta_1&,
\end{aligned}
$$

$$
\begin{aligned}
(T_2'.T_3' - N_1'.T_1') &= (N_2N_3 - T_1^2).\cos\alpha_2.\cos\alpha_3 \\
&+ (N_3N_1 - T_2^2).\cos\beta_2.\cos\beta_3 \\
&+ (N_1N_2 - T_3^2).\cos\gamma_2.\cos\gamma_3 \\
+ (T_2T_3 - N_1T_1)&.(\cos\beta_2.\cos\gamma_3 + \cos\gamma_2.\cos\beta_3) \\
+ (T_3T_1 - N_2T_2)&.(\cos\beta_1.\cos\gamma_3 + \cos\gamma_1.\cos\beta_3) \\
+ (T_1T_2 - N_3T_3)&.(\cos\beta_1.\cos\gamma_2 + \cos\beta_2.\cos\gamma_1).
\end{aligned}
$$

I think the theory of distortions has not been sufficiently investigated. I will give a short sketch of a theory of the subject. Denote by ξ, η, ζ the absolute displacements of a molecule. Suppose we change our coordinate axes and write

$$
\begin{aligned}
\xi' &= \cos\alpha_1.\xi + \cos\beta_1.\eta + \cos\gamma_1.\zeta, \\
\eta' &= \cos\alpha_2.\xi + \cos\beta_2.\eta + \cos\gamma_2.\zeta, \\
\zeta' &= \cos\alpha_3.\xi + \cos\beta_3.\eta + \cos\gamma_3.\zeta, \\
x' &= \cos\alpha_1.x + \cos\beta_1.y + \cos\gamma_1.z, \\
y' &= \cos\alpha_2.x + \cos\beta_2.y + \cos\gamma_2.z, \\
z' &= \cos\alpha_3.x + \cos\beta_3.y + \cos\gamma_3.z,
\end{aligned}
$$

but we know that x', y', z', and $\frac{d}{dx'}$, $\frac{d}{dy'}$, $\frac{d}{dz'}$, when the transformation of coordinates is orthogonal, are transformed by cogredient transformation, and therefore we get

$$
\frac{d\xi'}{dx'} = \left(\cos\alpha_1\frac{d}{dx} + \cos\beta_1\frac{d}{dy} + \cos\gamma_1\frac{d}{dz}\right)(\cos\alpha_1\xi + \cos\beta_1\eta + \cos\gamma_1\zeta),
$$

&c.,

$$
\frac{d\eta'}{dz'} + \frac{d\zeta'}{dy'}
$$

$$
= \left(\cos\alpha_3\frac{d}{dx} + \cos\beta_3\frac{d}{dy} + \cos\gamma_3\frac{d}{dz}\right).(\cos\alpha_2.\xi + \cos\beta_2.\eta + \cos\gamma_2.\zeta),
$$

$$
+ \left(\cos\alpha_2\frac{d}{dx} + \cos\beta_2\frac{d}{dy} + \cos\gamma_2\frac{d}{dz}\right).(\cos\alpha_3.\xi + \cos\beta_3.\eta + \cos\gamma_3.\zeta).
$$

I shall write for shortness and elegance

$$\frac{d\xi}{dx} = \Delta_1, \quad \frac{d\eta}{dy} = \Delta_2, \quad \frac{d\zeta}{dz} = \Delta_3,$$

$$\tfrac{1}{2}\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right) = \Gamma_1, \quad \tfrac{1}{2}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right) = \Gamma_2, \quad \tfrac{1}{2}\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right) = \Gamma_3;$$

therefore our formulæ of transformation give us

$$\begin{aligned} \Delta_1' = {} & \cos^2\alpha_1\Delta_1 + \cos^2\beta_1\Delta_2 + \cos^2\gamma_1\Delta_3 \\ & + 2.\cos\beta_1.\cos\gamma_1.\Gamma_1 \\ & + 2.\cos\alpha_1.\cos\gamma_1.\Gamma_2 \\ & + 2.\cos\alpha_1.\cos\beta_1.\Gamma_3.\ \Delta_2' = \&c., \end{aligned}$$

but these are identically the same formulæ of transformation for $\Delta_1'\Delta_2'\Delta_3'\Gamma_1'$, &c. as those before found for $N_1'N_2'N_3'T_1'$, &c.

Now imagine a plane through the coordinate centre, two mutually perpendicular lines in it and the axis of the plane, we will investigate the *delta* and two *gamma's* of this plane and their connection; multiply the values of $\Delta_1'\Gamma_3'\Gamma_2'$ in terms of $\Delta_1\Delta_2\Delta_3\Gamma_1\Gamma_2\Gamma_3$ by $\cos\alpha_1$, $\cos\alpha_2$, $\cos\alpha_3$ respectively, and add results bearing in mind that

$$\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1,$$

and $\quad \cos\alpha_1 . \cos\alpha_2 + \cos\beta_1 . \cos\beta_2 + \cos\gamma_1 . \cos\gamma_2 = 0$, &c.,

hence we get

$$\cos\alpha_1\Delta_1' + \cos\alpha_2\Gamma_3' + \cos\alpha_3\Gamma_2' = \cos\alpha_1\Delta_1 + \cos\beta_1\Gamma_3 + \cos\gamma_1\Gamma_2,$$

and similarly, we obtain

$$\cos\beta_1\Delta_1' + \cos\beta_2\Gamma_3' + \cos\beta_3\Gamma_2' = \cos\alpha_1\Gamma_3 + \cos\beta_1\Delta_2 + \cos\gamma_1\Gamma_1,$$

$$\cos\gamma_1\Delta_1' + \cos\gamma_2\Gamma_3' + \cos\beta_3\Gamma_2' = \cos\alpha_1\Gamma_2 + \cos\beta_1\Gamma_1 + \cos\gamma_1\Delta_3.$$

Imagine now a line drawn through the origin of coordinates of such a length and in such a direction that if it be projected on the axis of our ideal plane and the two rectangular lines in plane, components are $\Delta_1'\Gamma_3'\Gamma_2'$; therefore, evidently $\cos\alpha_1\Delta_1' + \cos\alpha_2\Gamma_3' + \cos\alpha_3\Gamma_2'$, &c. are the projections of this line on the coordinate axes; write them equal respectively to x, y, and z; therefore

$$x = \cos\alpha_1 . \Delta_1 + \cos\beta_1 . \Gamma_3 + \cos\gamma_1 . \Gamma_2,$$

$$y = \cos\alpha_1 . \Gamma_3 + \cos\beta_1 . \Delta_2 + \cos\gamma_1 . \Gamma_1,$$

$$z = \cos\alpha_1 . \Gamma_2 + \cos\beta_1 . \Gamma_1 + \cos\gamma_1 . \Delta_3,$$

but these equations are the symbolical expressions of a theorem analogous to the "tetrahedron of elasticity," this theorem we shall therefore call the "tetrahedron of distortion," setting out therefore from this theorem an analysis identically the same as the one conducted by us relative to the "tetrahedron of elasticity" leads to similar results, theorem for theorem. Not to trespass on this *Journal*, I shall merely state them, all of which are new, as far as I am aware.

(1) A plane whose *delta* and *gamma's* are $\Delta_1\Gamma_2\Gamma_3$ and the resultant of $\Delta_1\Gamma_2\Gamma_3$ has the radius vector and its conjugate diametral plane in a certain ellipsoid for their respective positions, and the length of this resultant is as $p\times r$, this ellipsoid we shall call the "ellipsoid of distortion."

(2) In a solid body there always exist three mutually perpendicular lines, the angles between which are still right after distortion, *i.e.* $\Gamma_1=0$, $\Gamma_2=0$, $\Gamma_3=0$.

(3) The "ellipsoid of distortion" is capable of being written in the forms

$$(\Delta_2\Delta_3-\Gamma_1^2).x^2+(\Delta_3\Delta_1-\Gamma_2^2).y^2+(\Delta_1\Delta_2-\Gamma_3^2).z^2$$
$$+2(\Gamma_2\Gamma_3-\Delta_1\Gamma_1).yz+2(\Gamma_3\Gamma_1-\Delta_2\Gamma_2)\,xz+2(\Gamma_1\Gamma_2-\Delta_3\Gamma_3)\,xy=\text{constant},$$

$$\frac{\Delta_1}{R_1^2}x^2+\frac{\Delta_2}{R_2^2}y^2+\frac{\Delta_3}{R_3^2}z^2+\frac{2\Gamma_1}{R_2R_3}yz+\frac{2\Gamma_2}{R_1R_3}xz+\frac{2\Gamma_3}{R_1R_2}xy=\text{constant},$$

$$\frac{x^2}{D_1}+\frac{y^2}{D_2}+\frac{z^2}{D_3}=\text{constant},$$

where D_1, D_2, and D_3 are values of Δ_1, when Γ_1, Γ_2, and Γ_3 each equal cypher.

(4) The three expressions, viz.

$$\Delta_1+\Delta_2+\Delta_3,$$
$$\Delta_2\Delta_3+\Delta_3\Delta_1+\Delta_1\Delta_2-\Gamma_1^2-\Gamma_2^2-\Gamma_3^2,$$
$$\Delta_1\Delta_2\Delta_3+2\Gamma_1\Gamma_2\Gamma_3-\Delta_1\Gamma_1^2-\Delta_2\Gamma_2^2-\Delta_3\Gamma_3^2$$

are invariant, that $\Delta_1+\Delta_2+\Delta_3$ is an invariant of course is otherwise self-evident, the third will of course seldom enter into the construction of the equations of equilibrium or motion of the solid, but we may use the second as Mr. Haughton has used the first (*Trans. R. I. A.*, Vol. XXII.).

(5) Calling the resultant R of $\Delta_1\Gamma_2\Gamma_3$ the distortion of plane for shortness, we get this theorem of "reciprocity of

distortion," viz. the distortion of a plane P projected on the axis of another plane P' equals distortion of P' projected on axis to P.

(6) The distortions Δ_1, Δ_2, Δ_3 are as squares of perpendiculars on tangent planes to "ellipsoid of distortion" parallel to their respective distortion planes.

Other theorems of course may similarly be deduced, but I conclude with a few remarks; and first, as the coefficients of the ellipsoid of distortion are $\Delta_2 . \Delta_3 - \Gamma_1^2$, &c., and of its reciprocal Δ_1, &c., we immediately perceive the truth of Mr. Haughton's remark (*Cambridge and Dublin Mathematical Journal*, Vol. IX., p. 131) that Δ_1, &c. and $\Delta_2\Delta_3 - \Gamma_1^2$, &c. are cogrediently transformed.

Secondly, if the body is homogeneous, it is manifest by the principle of sufficient reason that the principal axes of distortion coincide with principal axes of elasticity, but not if it be heterogeneous; therefore we obtain the theorem noticed by Mr. Rankine (*Cambridge and Dublin Mathematical Journal*, Vol. VI., p. 55) "In an elastic substance which is homogeneous and symmetrical with respect to molecular action there are three directions at right angles to each other, in which a longitudinal strain produces an exactly normal pressure on a plane at right angles to the direction of the strain." I shall only remark, in conclusion, that calling our three "distortion invariants" I_1, I_2, and I_3, if we suppose the sum of internal moments represented by the variation of a single function and body homogeneous and non-crystalline, this distortion theory gives us very simply the form of V, since it must be a symmetrical function of the second order of $\Delta_1\Delta_2$, &c. and must remain the same after transformation; therefore it must be a function of the invariants I_1 and I_2; therefore

$$= A . I_1^2 + B . I_2 .$$

Many other theorems and demonstrations may also be deduced by the "ellipsoid of distortion."

38, Trinity College, Dublin,
April 22, 1861.

ON COAXIAL CIRCLES.

By JOHN CASEY, Scholar of Trinity College, Dublin, and Head-Master of the National District Model School, Kilkenny.

(Continued from p. 53.)

14. IF in the last Article the arcs AD and BC be bisected it is easy to see that the anharmonic ratio of the four points of bisection on BC is equal to the anharmonic ratio of the four t's, hence the anharmonic ratio of the four points of bisection on the arc BC is equal to the anharmonic ratio of the corresponding points on the arc AD. *And therefore if the arcs* BC *and* AD *be bisected, the envelope of the line joining the point of contact is a conic, having double contact with the circle* X. Q.E.D.

15. ABCD (*fig.* 6) *is a quadrilateral such as described in Arts.* 13, 14, *and* abcd *the line of contacts,* AB *and* CD *intersect in* O′, *and* AC, BD *in* O, *the line* OO′ *passes through the point of contact of* ad *with its envelope.*

Demonstration. Let OO' intersect ad in l, and through O draw $a'd'$ parallel to ad. By similar triangles

$$al : ld :: a'O : Od',$$

and since Oa' bisects the angle AOB, and the triangles AOB and COD are similar,

$$a'O : Od' :: AB : CD;$$

therefore $al : ld :: AB : CD ::$ by Lemma 2, Cor. 2, velocity of the point a : velocity of the point d. Again, let e' be the point of contact of ad with its envelope, we have velocity of a : velocity of $d :: ae' : e'd$. Since e' is the instantaneous centre of rotation of ad, and the tangents at a and d make equal angles with ad, hence the points l and e' coincide. Q.E.D.

16. B, C (*fig.* 7) *are two fixed points on a circle* X *of a coaxial system, and* Bc, Cb *are tangents to a variable circle* Z *of the system, the locus of the points of contact* e *on each chord* bc *when it touches its envelope, generated by supposing* BC *to roll on a circle of the system, is a conic passing through the limiting points.*

Demonstration. Let AD and BC meet in O''. Now AD and BC are tangents to a circle X' of the system, Lemma 1, Cor. 1, and bc passes through the points of contact f, g. Now the triangle $OO'O''$ is self-conjugate with respect to X, hence O'' is the pole of OO' with respect to X, therefore the polar of O'' with respect to every circle of the system passes through some point on the line OO'. Again, fg is the polar of O'' with respect to X'; therefore the polar of O'' with respect to every circle of the system passes through e, hence the circle described on $O''e$ as diameter passes through the limiting points. Now while BC remains fixed, if we take four successive circles Z, we shall have corresponding to them four points O'' on the line BC produced, and four points e on the chords bc. Join the four O'''s to the limiting points, and also the four e's, and it is easy to see that the anharmonic ratio of the two pencils from the four e's to the limiting points are equal; hence the locus of e is a conic through the limiting points.

COR. *It is not difficult to prove that the vertices of the triangle* OO'O'', *the limiting points and the point where* BC *touches its envelope are on the same conic.*

17. In the last Article we see that the anharmonic ratio of the four points e is equal to the anharmonic ratio of the four points O'', and that again equal to the anharmonic ratio of the four lines gf, which is equal to the anharmonic ratio of the four lines tm drawn from t the point of contact on BC, with its envelope perpendicular to the chords bc, and since each tm passes through a focus of its reciprocant; hence we have the anharmonic ratio of the four foci of the conics equal to the anharmonic ratio of the four points e. Therefore we have the following theorem:

If a variable chord BC *of a circle of a coaxial system touch another circle of the system, and from its extremities tangents be drawn to four other circles of the system, then in each position of* BC *the points where the cords touch their envelope lie on a conic* (which we may call the transversal conic) *passing through the limiting points, and the anharmonic ratio of the four points in which any transverse conic is cut by the four reciprocants is equal to that of the points in which any other transversal conic is cut by the same reciprocants.*

COR. *It is evident also that the anharmonic ratio of the four points in which any reciprocant is cut by four transversal*

conics is equal to that of the four points in which any other reciprocant of the system is cut by the same transversals.

18. We have seen in Art. 4, that O, O' being the centres of the circles Y and Z, the reciprocant of a circle X, whose centre is S, with respect to Y and Z, will be a parabola when the square of the radius of X is equal to $SO.SO'$. By examining this particular case we shall obtain some interesting results.

Since the square of the radius of $X = SO.SO'$, X is the circle described on the distance between the centre of similitude of Y and Z, and since the foci of the reciprocant divide OO' harmonically, and one of them is at infinity, the other bisects OO'. Again, because the circle on the transverse is one of the circles of the system, it is plain that the vertex of the parabola touches the radical axis. Thirdly. If α' and β' denote the complementary and sub-complementary of Y and Z with respect to X, one of these circles will pass through the focus of the parabola, and the other will be the radical axis. This follows from the property that the foci of the reciprocant are centres of similitude of the circle on its transverse, and the complementary and sub-complementary respectively.

19. By means of parabolic properties we can obtain theorems respecting X, Y, and Z.

1°. The circle described about the triangle formed by three tangents to a parabola passes through its focus. Hence *if from a point* A *in* X, *tangents* AB, AB′, AC, AC′ *be drawn to* Y *and* Z *respectively, the circle described about any of the four triangles formed by the lines* BC, B′C′, B′C BC′ *will pass through a given point wherever the point* A *is taken on the circumference of* X.

2°. Since the perpendiculars of a triangle whose sides touch a parabola intersect on its directrix, the perpendiculars of the four triangles formed by the lines BC, $B'C$, BC', $B'C'$ intersect on the directrix, and this line is the radical axis of the circles on the diagonals of the quadrilateral formed by these lines. Hence we have the following theorem: *If from any point on the circumference of the circle described on the distance between the centres of similitude of two circles of two given circles a pair of tangents be drawn to each the radical axis of the circles described on the chords of contact is a fix d line.*

3°. The following theorem is also easily inferred: *The parabola which touches the three sides of a triangle, and whose focus is either extremity of any of the three diameters which are perpendicular respectively to the sides of the triangle has a fourth tangent which is also a common tangent to two of the four circles which touch the three sides of the triangle.*

TRIANGULAR SYSTEMS OF CIRCLES.

20. *If* ABC, A'B'C' *be two Poncelet's triangles inscribed in a circle* X, *let the circles which are the envelopes of these triangles opposite the angles* A, B, C, A', B', C', *respectively be denoted by* a, b, c, α, β, γ, *then if* α', β', γ' *form triangular systems with* a, α; b, β; c, γ, *respectively*, α', β', γ' *form a triangular system.*

Demonstration. Let the triangle $A'B'C'$ (fig. 8) be turned round until the point A' coincide with A, and let its position then be $AB'C'$. Now it is evident that the chord BB' touches γ', and that CC' touches β', from C' draw $C'C''$ touching γ', and join BC''. Then BC'', per Lemma 1, touches the same circle as $B'C'$; therefore BC'' touches α, and BC touches a; therefore CC'' touches α', and the three sides of the triangle $CC'C''$ touch the circles α', β', γ' respectively. Q.E.D.

21. *If a Poncelet's hexagon inscribed in a circle* X *be such that the circles which are the envelopes of three alternate sides* AB, CD, EF *form a triangular system with respect to* X. *The circles which are the envelopes of the three alternate sides* BC, DE, FA *also form a triangular system.* This is evident from Art. 20, since the envelopes of AC, CE, EA form a triangular system. Q.E.D.

22. *If the circles* Y, Z, W *form a triangular system with respect to* X, *and from any point in* X *a pair of tangents be drawn to each, intersecting* X *again the points* A, A'; B, B'; C, C', *the envelopes of the chords* AA', BB', CC' *form a triangular system of circles with respect to* X.
This is evident from Art. 20.

23. *If* Y, Z, W *be three circles forming a triangular system with respect to* X, *then it follows from Art.* 20, *that if we take the duplicate of each with respect to the other two, the three duplicates form a triangular system of circles with respect to* X. This is evident from Art 20.

24. It is not difficult to extend the results of Arts. 20, 21, 22 to any polygon whatever.

THEOREMS ON RECIPROCATION.

25. Let $ABCD$ inscribed in X have its sides AB, CD, (fig. 7) tangents to a given circle Y, and AC, BD tangents to another given circle Z of a coaxial system; let the sides of this quadrilateral intersect in the points O, O', O''. Now it has been proved that the polar of the point e with respect to any circle of the system passes through O'', and it is evident that the polar of e with respect to Z passes through O; therefore OO'' is the polar of e with respect to Z, and O is the pole of bc with respect to Z. Now the locus of e is the envelope of bc; therefore the locus of O is the envelope of OO''; in like manner the locus of O' is the envelope of $O'O''$. Again, since the locus of e is the reciprocal of X with respect to Y and Z; let this reciprocant be denoted by Σ, then it is evident that the locus of O' is the reciprocal of Σ with respect to Y, and the locus of O is the reciprocal of the locus of O' with respect to X, since $OO'O''$ is a self-conjugate triangle with respect to X, and finally, if the locus of O be reciprocated with respect to Z, it will reproduce Σ back again. Hence we have the following theorem. *If Σ be the reciprocant of a circle* X *of a coaxial system with respect to two other circles* Y *and* Z *of the system, then if Σ be reciprocated with respect to* Y, *the result reciprocated with respect to* X, *and lastly, this result reciprocated with respect to* Z, *Σ will be the result of these three successive reciprocations.* Q.E.D.

26. We may express the theorem in the last Article by the following notation,

$$ZXY.\Sigma = \Sigma.$$

Where Σ is the subject, Z, X, Y, beginning with Y, are the symbols of reciprocation, the meaning of which is that we first reciprocate Σ with respect to Y, then the result with respect to X, and finally this result with Z, and we get Σ itself as the result of these successive reciprocations performed upon Σ, and the dot is placed between the operator and subject.

Cor. It will appear in like manner, that

$$YXZ.\Sigma = \Sigma.$$

27. If we examine the demonstration of Art. 25 attentively, we shall find that the following proposition is contained in it; let e be any point on Σ, and bc the tangent to Σ at e, then

$$YXZ.e = bc.$$

That is, if we take the polar of e *with respect to* Z, *then the pole of this polar with respect to* X, *and lastly, the polar of this pole will be* bc *the tangent to* Σ *at the point* e.

In like manner,

$$ZXY.e \;= bc,$$
$$YXZ.bc = e,$$
$$ZXY.bc = e.$$

28. The propositions contained in the two last Articles are but particular cases of more general ones which we proceed to develop.

Let abc (fig. 9) be any triangle circumscribing Σ {we omit Y, X, Z in the diagram for the sake of distinctness}. Then since $YXZ.ab = e$, $YXZ.ac = e'$, and $YXZ.bc = e''$, it is evident that the operation $YXZ.$ performed on the triangle abc gives the triangle $ee'e''$ as the result. Again, if the locus of the point a be the line BC, the line cc' will pass through O the pole of BC with respect to Σ, hence the operation $YXZ.$ performed on BC gives as result the point O the pole of BC with respect to Σ. *Therefor, if upon any point the four successive reciprocations denoted by* YXZΣ. *be performed, the result will be the point itself.* Q.E.D.

29. If we take any figure whatever and perform the reciprocations denoted $YXZ\Sigma.$ upon any point in it the result will be that identical point, and therefore when the operations $YXZ\Sigma$ are performed upon the whole figure, the result will be the whole figure itself; hence, let S denote any figure we have,

$$YXZ\Sigma.S = S,$$
$$ZXY\Sigma.S = S.$$

30. From the result of the last Article many important inferences can be deduced, for doing which it will be useful to bear in mind the following principle, viz.: *If in any reciprocating symbol the same letter occur twice in succession, such as* ZZ, *it may be omitted.*

For the operation ZZ performed upon any point or line gives as result that identical point or line; therefore when performed upon any figure it gives as result the figure itself.

31. In the formula $YXZ\Sigma.S$, replace S by $\Sigma.S$, and we have

$$YXZ\Sigma\Sigma.S=\Sigma.S,$$

or $$YXZ.S=\Sigma.S, \text{ Art. 30.}$$

Hence, *if* Σ *be the reciprocant of* X *with respect to* Y *and* Z, *the result of performing the three successive reciprocations denoted by* YXZ *upon any figure is the reciprocal of the figure with respect to* Σ.

32. Upon the result contained in the last Article perform the equivalent operations $YXZ.$ and Σ, and we have

$$(YXZ)^2.S=\Sigma\Sigma.S=S,$$

where the index denotes that the operation is performed twice in succession. *Hence we have the following theorem: If* Y, X, Z *denote any three coaxial circles, the operation* YXZ *performed twice in succession upon any figure gives as result that figure itself.*

Cor. *All odd powers of the operation* XYZ *are equal, and all even powers produce the original figure.*

33. From the last Article, we have

$$(YXZ)^2.S=S,$$

$$(YX'Z)^2.S=S.$$

Hence $$(YXZ)^2.=(YX'Z)^2.,$$

or omitting the subject,

$$YXZYXZ.=YX'ZYX'Z.,$$

and omitting the first and last letters from each,

$$XZYX=X'ZYX'.$$

Therefore if ZY. *be any reciprocating symbol, we get the same result, whatever letter we place before and after it.*

34. In the reciprocating equation

$$XZYX=X'ZYX',$$

replace X' by Y, and we have, by Art. 30,

$$XZYX = YZ.$$

Hence, if we place any letter before and after a binomial symbol ZY, *the result will be equal to the operation* ZY *reversed.*

35. In the equation $XZYX. = YZ$ of Art. 34, place W before and after the equalities on both sides, and we get

$$WXZYXW = WYZW = ZY, \text{ by Art. 34.}$$

Hence two letters before any binomial symbol are cancelled by placing the same letters reversed after.

36. If in the equation

$$ZXY\Sigma . S = S,$$

we replace S by $\Sigma Y. S$, we have

$$ZXY\Sigma\Sigma Y. S = \Sigma Y. S,$$

or omitting $\Sigma\Sigma$ in the left-hand side of the equation, and then YY, Art. 30,

$$ZX. S = \Sigma Y. S.$$

Hence, *in a reciprocating equation, when two or more members are removed from one side to the other, their order must be reversed.*

37. In the last equation let Z and Y become equal, then Σ becomes the reciprocal of X with respect to Y, let it be denoted by Σ_1, and we have

$$YX. S = \Sigma_1 Y. S.$$

Hence, *if* X *and* Y *be any two circles, and if we reciprocate any figure first with respect to* X, *and then with respect to* Y, *the result will be the same as when we reciprocate with* Y *and then* Σ_1, *the reciprocal of* X *with respect to* Y.

38. In the last equation replace S by Y, and denoting the reciprocal of Y with respect to X by Σ_2, we have

$$Y. \Sigma_2 = \Sigma_1 Y,$$

in like manner, $$X. \Sigma_1 = \Sigma_2 . X.$$

Hence, *if* X *and* Y *be any two circles, and* Σ_1 *and* Σ_2 *the reciprocals of* X *and* Y *with respect to* Y *and* X *respectively, the reciprocal of* Σ_1 *with respect to* X *is also the reciprocal of* X *with respect to* Σ_2.

39. In the equation $ZX.S=\Sigma Y.S$ of Art. 36, replace S by X, and we have

$$ZX.X=\Sigma Y.X,$$

or

$$Z.X=\Sigma Y.X.$$

Hence we have the following theorem:

If Σ *be the reciprocant of* X *with respect to* Y *and* Z, *and* S_1 *and* S_2 *the reciprocals of* X *with respect to* Y *and* Z *respectively, then* S_1 *is the reciprocal of* S_2 *with respect to* Σ.

From Art. 4 the following relation between the conics S_1, S_2, and Σ is easily inferred, viz.: *the eccentricity of* Σ *is a mean proportional between the eccentricities of* S_1 *and* S_2.

40. We shall conclude this paper by giving a geometrical proof of the theorem of Art. 39.

Let X, Y, and Z be the three circles; from any point A in X draw the tangents AB, AB' to Y, and AC, AC' to Z. Join BC, BC', $B'C$, $B'C'$, then four lines are evidently tangents to Σ, and BB' and CC' are tangents to S_1 and S_2 respectively; let the points of contact on BC, $B'C$, and BB' be denoted by P, P', P'', and $B'C'$, BC', and CC' by P_1, P_1', P_1'' respectively.

Now it is easily proved by infinitesimals, that

$$AB.BP : AC.CP :: \text{velocity of } B : \text{velocity of } C,$$

by considering P as the instantaneous centre of rotation of BC, and in like manner,

$$AC.CP' : AB'.B'P' :: \text{velocity of } C : \text{velocity of } B',$$

$$B'P'' : BP'' :: \text{velocity of } B' : \text{velocity of } B.$$

Hence, by compounding these ratios, and since $AB=AB'$, we get

$$BP.CP'.B'P''=CP.B'P'.BP'';$$

therefore the points P, P', P'' are in a right line, in like manner P_1, P_1', P'' are in a right line, also P, P_1', P_1'' are in a right line, and P_1, P', P_1'' are in a right line, and it is easily seen from this, that P'' is the pole of CC' with respect to Σ, and BB' the polar of P_1''. Hence the proposition is evident. Q.E.D.

Cor. *Hence the reciprocant is real even when the tangents* AB, AB′, AC, AC′ *are imaginary.*

Model School, Kilkenny,
Nov. 29, 1860.

ON CERTAIN ANALYTICAL RELATIONS BETWEEN CONJUGATE WAVE-VELOCITIES, RAY-VELOCITIES, AND PLANES OF POLARIZATION.

By WILLIAM WALTON, M A, Trinity College.

THE position of each of two conjugate planes of polarization in a biaxial crystal, as well as the direction and magnitude of each of the two corresponding ray-velocities, may be shewn to depend upon two variables, viz., the two conjugate wave-velocities. The formulæ, by which these relations may be expressed, are rather elegant, and may perhaps be worth recording as a contribution to the Geometry of the Wave-Surface.

Let v represent any wave-velocity, and l, m, n, the cosines of its inclination to the axes of elasticity: let α, β, γ, denote the direction-cosines of the corresponding plane of polarization, and r the corresponding ray-velocity. Let x, y, z, be the projections of r on the axes of elasticity.

Then, (see Griffin's *Treatise on Double Refraction*, pp. 11, 18)

$$\frac{l}{\alpha(v^2-a^2)} = \frac{m}{\beta(v^2-b^2)} = \frac{n}{\gamma(v^2-c^2)} \quad \text{.........(1)},$$

and

$$\left.\begin{aligned} \frac{v^2-a^2}{r^2-a^2}\,x &= vl \\ \frac{v^2-b^2}{r^2-b^2}\,y &= vm \\ \frac{v^2-c^2}{r^2-c^2}\,z &= vn \end{aligned}\right\} \quad \text{.....................(2)}.$$

Eliminating l, m, n, between (1) and (2), we get

$$\frac{x}{\alpha(r^2-a^2)} = \frac{y}{\beta(r^2-b^2)} = \frac{z}{\gamma(r^2-c^2)}:$$

ence, by the equation to the Wave-Surface, viz.:

$$\frac{a^2x^2}{r^2-a^2} + \frac{b^2y^2}{r^2-b^2} + \frac{c^2z^2}{r^2-c^2} = 0,$$

we have

$$a^2\alpha^2 (r^2 - a^2) + b^2\beta^2 (r^2 - b^2) + c^2\gamma^2 (r^2 - c^2) = 0,$$

$$r^2 = \frac{a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2}{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2} \quad \text{..................(3).}$$

Since the sum of the squares of α, β, γ, is equal to unity, we have, by the equations (1),

$$\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} = \frac{l^2}{\alpha^2 (v^2 - a^2)^2},$$

and therefore

$$\alpha^2 = \frac{\dfrac{l^2}{(v^2 - a^2)^2}}{\dfrac{l^2}{(v^2 - a^2)^2} + \dfrac{m^2}{(v^2 - b^2)^2} + \dfrac{n^2}{(v^2 - c^2)^2}} \quad \text{.........(4).}$$

Let α_1, α_2, be the two values of α, and v_1, v_2, the two values of v, which correspond to any assigned values of l, m, n: then, (see Griffin's *Double Refraction*, p. 14),

$$\left.\begin{aligned} l^2 (a^2 - b^2)(a^2 - c^2) &= (v_1^2 - a^2)(v_2^2 - a^2) \\ m^2 (b^2 - c^2)(b^2 - a^2) &= (v_1^2 - b^2)(v_2^2 - b^2) \\ n^2 (c^2 - a^2)(c^2 - b^2) &= (v_1^2 - c^2)(v_2^2 - c^2) \end{aligned}\right\} \quad \text{....... (5).}$$

From (4) and (5), using Σ to denote symmetrical summation, we see that

$$\alpha_1^2 = \frac{(v_2^2 - a^2)(v_1^2 - b^2)(v_1^2 - c^2)(b^2 - c^2)}{\Sigma \{(v_2^2 - a^2)(v_1^2 - b^2)(v_1^2 - c^2)(b^2 - c^2)\}}.$$

Now it may easily be ascertained that the sum of the coefficients of v_2^2 in the denominator is equal to

$$-(b^2 - c^2)(c^2 - a^2)(a^2 - b^2),$$

and that the sum of all the terms, which do not involve v_2^2, is equal to

$$(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)\, v_1^2:$$

hence we have

$$\alpha_1^2 = \frac{(v_2^2 - a^2)(v_1^2 - b^2)(v_1^2 - c^2)}{(c^2 - a^2)(a^2 - b^2)(v_1^2 - v_2^2)}.$$

Similarly,
$$\beta_1^2 = \frac{(v_2^2 - b^2)(v_1^2 - c^2)(v_1^2 - a^2)}{(a^2 - b^2)(b^2 - c^2)(v_1^2 - v_2^2)},$$

$$\gamma_1^2 = \frac{(v_2^2 - c^2)(v_1^2 - a^2)(v_1^2 - b^2)}{(b^2 - c^2)(c^2 - a^2)(v_1^2 - v_2^2)}.$$

These expressions for α_1, β_1, γ_1, determine the position of one of the planes of polarization in terms of the two conjugate wave-velocities. Interchanging the suffixes, we obtain the expressions for determining the position of the other plane of polarization.

Again, let r_1 be the value of r corresponding to the value v_1 of v: then, from (3),

$$r_1^2 = \frac{a^4\alpha_1^2 + b^4\beta_1^2 + c^4\gamma_1^2}{a^2\alpha_1^2 + b^2\beta_1^2 + c^2\gamma_1^2},$$

or, substituting for α_1, β_1, γ_1, the values obtained above,

$$r_1^2 = \frac{\Sigma\left\{a^4(b^2-c^2)\dfrac{v_2^2-a^2}{v_1^2-a^2}\right\}}{\Sigma\left\{a^2(b^2-c^2)\dfrac{v_2^2-a^2}{v_1^2-a^2}\right\}}.$$

Now the numerator of this fraction is equal to

$$(v_2^2-v_1^2)\,\Sigma\left\{\frac{a^4(b^2-c^2)}{v_1^2-a^2}\right\} + \Sigma\{a^4(b^2-c^2)\}$$

$$= (v_2^2-v_1^2)\,\Sigma\left\{(b^2-c^2)\left(\frac{a^2v_1^2}{v_1^2-a^2}-a^2\right)\right\} - (b^2-c^2)(c^2-a^2)(a^2-b^2)$$

$$= v_1^2(v_2^2-v_1^2)\,\Sigma\left\{\frac{a^2(b^2-c^2)}{v_1^2-a^2}\right\} - (b^2-c^2)(c^2-a^2)(a^2-b^2)$$

$$= v_1^4(v_2^2-v_1^2)\,\Sigma\left\{\frac{b^2-c^2}{v_1^2-a^2}\right\} - (b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Again, the denominator of the fraction is equal to

$$(v_2^2-v_1^2)\,\Sigma\left\{\frac{a^2(b^2-c^2)}{v_1^2-a^2}\right\} = v_1^2(v_2^2-v_1^2)\,\Sigma\left\{\frac{b^2-c^2}{v_1^2-a^2}\right\}.$$

Hence $r_1^2 = v_1^2 - \dfrac{1}{v_1^2(v_2^2-v_1^2)} \cdot \dfrac{(b^2-c^2)(c^2-a^2)(a^2-b^2)}{\Sigma\left\{\dfrac{b^2-c^2}{v_1^2-a^2}\right\}}.$

But $$\Sigma\left\{\frac{b^2-c^2}{v_1^2-a^2}\right\} = -\frac{(b^2-c^2)(c^2-a^2)(a^2-b^2)}{(v_1^2-a^2)(v_1^2-b^2)(v_1^2-c^2)}:$$

hence $$r_1^2 = v_1^2 + \frac{(v_1^2-a^2)(v_1^2-b^2)(v_1^2-c^2)}{v_1^2(v_2^2-v_1^2)}.$$

Similarly, $$r_2^2 = v_2^2 + \frac{(v_2^2-a^2)(v_2^2-b^2)(v_2^2-c^2)}{v_2^2(v_1^2-v_2^2)}.$$

These two formulæ determine the magnitudes of the ray-velocities in terms of the two conjugate wave-velocities.

Again, from the equations (2), x_1, y_1, z_1, being the projections of r_1 on the axes of elasticity, we have

$$x_1^2 = v_1^2 l^2 \left(\frac{r_1^2 - a^2}{v_1^2 - a^2}\right)^2$$

$$= v_1^2 \frac{(v_1^2 - a^2)(v_2^2 - a^2)}{(a^2 - b^2)(a^2 - c^2)} \cdot \left\{1 + \frac{(v_1^2 - b^2)(v_1^2 - c^2)}{v_1^2 (v_2^2 - v_1^2)}\right\}^2,$$

with analogous formulæ for y_1, z_1. The formulæ for x_2, y_2, z_2, are obtained from these by the interchange of suffixes.

These formulæ determine the directions of the ray-velocities in terms of the two conjugate wave-velocities.

Cor. Supposing the position of the plane front of a wave in a crystal to be assigned, the magnitudes of the two wave-velocities are determined by the formula

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0,$$

and therefore, by the formulæ above given for the values of $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, (x_1, y_1, z_1), (x_2, y_2, z_2), the positions of the two corresponding planes of polarization and the directions of the two corresponding rays may be computed.

March, 1861.

ABSTRACT OF SIR W. ROWAN HAMILTON'S EXPOSITION OF ABEL'S ARGUMENT.

By James Cockle.

In the following abstract of Sir W. R. Hamilton's paper on Abel's argument (*Trans. R. I. A.*, Vol. XVIII., pp. 171–259) I have made a brief statement of results suffice for those portions which, on a general view, coincide with the corresponding portions of Abel's discussion. But (excepting always Sir W. R. Hamilton's comprehensive and expressive notation) I have gone fully into that part on which, more particularly, there is a great diversity between the two arguments.

ABEL'S ARGUMENT, AS MODIFIED BY SIR W. R. HAMILTON.

[1.] That function which, according to a phraseology proposed by Abel, is called *an irrational function of the* μ^{th} *order*, and which will here be denoted by v (and into which no radicals with composite exponents are supposed to enter, since it is obvious that any extraction of a radical with a composite exponent may be reduced to a system of successive extractions with prime exponents), may be regarded as the general type of every conceivable function of any finite number of independent variables, which can be formed by any finite number of additions, subtractions, multiplications, divisions, elevations to powers, and extractions of roots of functions. Insomuch that the question, "Whether it be possible to express a root x of the general equation of the n^{th} degree,

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0,$$

in terms of the coefficients of that equation, by any finite combination of radicals and rational functions?" is, as Abel has remarked, equivalent to the question, "Whether it be possible to equate a root of the general equation of any given degree to an irrational function of the coefficients of that equation, which function shall be of any finite order μ?"

[2.] For the cases $n=2$, $n=3$, $n=4$, this question has long since been determined in the affirmative, by the discovery of the known solutions of the general quadratic, cubic, and biquadratic equations. (The solutions are then illustrated by translating them into functions of surds of different orders. I would add that the varied and important illustrations with which the paper here abstracted abounds are, I believe, chiefly or entirely due to Sir W. Rowan Hamilton.)

[3.] Inasmuch as by known processes we may (1) put the development of v under the form*

$$v = q_0 + q_1 p^{\frac{1}{n}} + q_2 p^{\frac{2}{n}} + \ldots + q_{n-1} p^{\frac{n-1}{n}},$$

$q_0, q_1, \ldots q_{n-1}$ being, in general, functions of the μ^{th} order,

* The n and m introduced for convenience into Arts. [3.] and [4.] are distinct from the n of the given equation and the m of order. The same remark holds of n and μ in Art. [5.]. I have ventured to transfer to [5.] the definition of irreducibility.—J. C.

not involving the radical $\sqrt[n]{(p)}$; and (2) the development of q_R under the form

$$q_R = r_0 + r_1 q^{\frac{1}{m}} + r_2 q^{\frac{2}{m}} + \ldots + r_{m-1} q^{\frac{m-1}{m}},$$

and so on; it is clear that by a repetition of this process v may be put under the form

$$v = \Sigma u p^{\frac{a}{n}} q^{\frac{b}{m}} \ldots,$$

in which 1°. u is a function of the order $\mu - 1$, or of a lower order; 2°. the exponent, a for example, is zero or any integer less than n; 3°. the summation extends to all the $n \times m \times \ldots$ terms which have the exponents $a, b, \ldots$ subject to the conditions just mentioned; and 4°. the coefficients u admit of being analogously developed. (This is illustrated by the cases of cubic and biquadratic equations).

[4.] Let v' be formed from v by changing every radical, such as $p^{\frac{1}{n}}$, to a corresponding product, such as $\omega_n p^{\frac{1}{n}}$, in which ω_n is an n^{th} root of unity; so that

$$v' = \Sigma u \omega_n{}^a \omega_m{}^b \ldots p^{\frac{a}{n}} q^{\frac{b}{m}} \ldots;$$

and let any isolated term of the corresponding development of v be denoted by

$$t_\nu = u_\nu p^{\frac{a}{n}} q^{\frac{b}{m}} \ldots;$$

we shall also have

$$t_\nu = \frac{1}{nm\ldots} \Sigma v' \omega_n{}^{-a} \omega_m{}^{-b} \ldots,$$

the summation extending as before, and also embracing all the $mn\ldots$ values of v (or v') which arise from the replacing of $\omega_n, \omega_m, \ldots$ by different $n^{\text{th}}, m^{\text{th}}, \ldots$ roots of unity. Sir W. R. Hamilton exemplifies this by the theory of cubics and biquadratics. In these examples the truth of the results is obvious; and the general demonstration follows easily from the properties of the roots of unity.

[5.] We may call an irrational function *irreducible*, when it is impossible to express that function, or any one of its component radicals, by any smaller number of extractions of prime roots of variables, than the number which the actual expression of that function or radical involves; even by introducing roots of constant quantities, or of numerical equa-

tions, which roots are in this whole discussion considered as being themselves constant quantities, so that they neither influence the order of an irrational function, nor are included among the radicals denoted by the symbols $\sqrt[n]{(p)}$, &c. We have hitherto made no use of the assumed *irreducibility* of the irrational function v. But now taking this property into account, we perceive that the radicals $p^{\frac{1}{n}}$, &c., or, as we may write them, z, &c., must not be compatible with any equations of condition whatever, but only with the equations of definition, such as

$$\{\sqrt[n]{(p)}\}^n = p,$$

which determine the radicals by determining their prime powers. The coexistence of

$$z^n - p = 0,$$

and
$$z^g + G_1 z^{g-1} + \ldots + G_{g-1} z + G_g = 0,$$

where g is less than n and $G_1, \ldots G_g$ are functions of orders not higher than that of z, and not involving z, is inconsistent with the irreducibility of z. And even if a system of relations

$$G_{g-e} = \nu_e z^e, \quad \text{or} \quad G_e = \nu_{g-e} z^{g-e},$$

in which ν_e is finite and a root of a numerical equation, were compatible with the equation of definition, so that the equation of condition became

$$(1 + \nu_{g-1} + \nu_{g-2} + \ldots + \nu_2 + \nu_1 + \nu_0)\, z^g = 0;$$

still, n being prime, we could find integers λ and μ satisfying the relation

$$\lambda n - \mu e = 1,$$

and, deducing successively,

$$z^{-\mu e} = \nu_e^{\ \mu} G_{g-e}^{\ -\mu}, \quad z^{\lambda n - \mu e} = z = \nu_e^{\ \mu} G_{g-e}^{\ -\mu} p^{\lambda},$$

we should either express z as a rational function of other radicals of the same system and of orders not higher than its own; or, raising both sides to the n^{th} power, we should be led to

$$p = (\nu_e^{\ \mu} G_{g-e}^{\ -\mu} p^{\lambda})^n,$$

an equation of condition, between the remaining radicals, which might be treated like the former, till at last an expression should be obtained of one of the remaining radicals as a rational function of other radicals of the

same system and of orders not higher than its own. In every case therefore we should be conducted to a diminution of the number of prime roots in v, which consequently would not be irreducible. A ready illustration of this is offered by the expression for the roots of a cubic.

[6.] It follows that if any *one* value of v be equal to any one root of

$$x^s + A_1 x^{s-1} + \ldots + A_{s-1} x + A_s = 0,$$

in which $A_1, \ldots A_s$ are any rational functions of $a_1, \ldots a_n$, then the same equation must be satisfied, also, for *all* values of v obtained by changing any radical z into ωz. And all the values of v must represent roots of the same s-ic equation, and, being *unequal* (otherwise an equation of condition, between the radicals, would arise inconsistent with irreducibility), must represent *different* roots of that s-ic.

[7.] Combining Arts. [4.] and [6.] we see that if any root x, of $x^s + \&c. = 0$, be equal to v, and v be developed under the form above assigned, then every term, say t_ν, of this development may be expressed as a rational (and indeed linear) function of some or all the s roots $x_1, x_2, \ldots x_s$ of the same proposed equation. Apt illustrations of this are found in the case of cubic, biquadratic, and quadratic equations.

[8.] To illustrate, by a preliminary example, the reasonings to which we are next to proceed, let it be supposed that any two of the terms t_ν are of the forms

$$t_1 = u_1 z_1^2 z_2 \zeta_1^3 \zeta_2^4,$$
$$t_2 = u_2 z_1 z_2 \zeta_1^2 \zeta_2^3,$$

in which the radicals are defined by equations such as the following:

$$z_1^3 = f_1, \quad z_2^3 = f_2, \quad \zeta_1^5 = g_1, \quad \zeta_2^5 = g_2,$$

their exponents being respectively equal to the numbers 3, 3, 5, 5, and the terms and radicals being all of the second order. We shall then have, by raising the two terms t to suitable powers, and attending to the equations of definition, the following expressions:

$$t_1^{10} = u_1^{10} f_1^6 f_2^3 g_1^6 g_2^8 z_1^2 z_2,$$
$$t_2^{10} = u_2^{10} f_1^3 f_2^3 g_1^4 g_2^6 z_1 z_2,$$
$$t_1^6 = u_1^6 f_1^4 f_2^2 g_1^3 g_2^4 \zeta_1^3 \zeta_2^4,$$
$$t_2^6 = u_2^6 f_1^2 f_2^2 g_1^2 g_2^3 \zeta_1^2 \zeta_2^3,$$

which give

$$T_1 = C_1 z_1, \quad T_2 = C_2 z_2, \quad T_3 = C_3 \zeta_1, \quad T_4 = C_4 \zeta_2,$$

if we put, for abridgment,

$$T_1 = \left(\frac{t_1}{t_2}\right)^{10}, \quad C_1 = \left(\frac{u_1}{u_2}\right)^{10} f_1^3 g_1^2 g_2^3,$$

$$T_2 = \left(\frac{t_2^2}{t_1}\right)^{10}, \quad C_2 = \left(\frac{u_2^2}{u_1}\right)^{10} f_2^3 g_1^2 g_2^4,$$

$$T_3 = \left(\frac{t_1^3}{t_2^4}\right)^6, \quad C_3 = \left(\frac{u_1^3}{u_2^4}\right)^6 f_1^4 f_2^{-2} g_1,$$

$$T_4 = \left(\frac{t_2^3}{t_1^2}\right)^6, \quad C_4 = \left(\frac{u_2^3}{u_1^2}\right)^6 f_1^{-2} f_2^3 g_2.$$

And, with a little attention, it becomes clear that the same sort of process may be applied to the terms of the development of any irreducible function v whatever its order; so that we have in general, a system of relations, such as the following:

$$T^{(m)} = C^{(m-1)} z^{(m)},$$

in which $T^{(m)}$ is of the order (say) m and is the product of certain powers (with exponents positive, negative, or null) of the various terms t_ν; $z^{(m)}$ is of the same order; and the coefficient $C^{(m-1)}$ is different from zero, but of an order lower than m. For if any radical of the order m were supposed to be so inextricably connected, in every term, with one or more of the remaining radicals of the same highest order, that it could not be disentangled from them by a process of the foregoing kind; and that thus the foregoing *analysis* of v should be unable to conduct to separate expressions for those radicals; it would then, reciprocally, have been unnecessary to calculate them separately, in effecting the *synthesis* of that function; which function, consequently, would not be irreducible. In the case of the lower equations the theorem of the present article is seen at once to hold good; because in these the radicals of the highest order are themselves terms of the development in question.

[9.] By raising to the proper powers the general expressions of the form just obtained, we obtain a system of, say, $n^{(m)}$ equations of this other form

$$\{T^{(m)}\}^{m'} = \{C^{(m-1)}\}^{m'} f = f',$$

f' being some new irrational function, of an order lower than m; and by combining the same expressions with those which define the various terms t, the number of which terms we shall denote by the symbol $t^{(m)}$, we obtain another system of $t^{(m)}$ equations, of which the following is a type:

$$U^{(m-1)} = w',$$

if we put, for abridgment,

$$U^{(m-1)} = t.\{T_1^{(m)}\}^{-\alpha}.\{T_2^{(m)}\}^{-\beta}\ldots,$$

and $$w' = u.\{C_1^{(m-1)}\}^{-\alpha}.\{C_2^{(m-1)}\}^{-\beta}\ldots.$$

In this manner we obtain in general $n^{(m)} + t^{(m)}$ equations, in each of which the product of certain powers (with positive, negative, or null exponents) of the $t^{(m)}$ terms of the development of v (or, as we may write it, $v^{(m)}$) is equated to some other irrational function, f' or w', of an order lower than m. Indeed it is to be observed, that since these various equations are obtained by an elimination of the $n^{(m)}$ radicals of highest order, between their $n^{(m)}$ equations of definition and the $t^{(m)}$ expressions for the $t^{(m)}$ terms of the development of $v^{(m)}$, they cannot be equivalent to more than $t^{(m)}$ distinct relations. But, among them, they must involve explicitly all the radicals of lower orders, which enter into the composition of the irreducible function $v^{(m)}$. For if any radical $z^{(k)}$, of order lower than m, were wanting in all the $n^{(m)} + t^{(m)}$ functions of the forms f' and w', we might then employ, instead of the old system of radicals, $z^{(m)}$ of the order m, a new and equally numerous system of radicals, $y^{(m)}$, according to the following type

$$y^{(m)} = T^{(m)} = \sqrt[\nu]{(\phi)};$$

and might then express all the $t^{(m)}$ terms of $v^{(m)}$ according to the formula

$$t = vy_1^{\frac{a}{n}}y_2^{\frac{b}{m}}\ldots,$$

or $$t = v\pi^{\frac{a}{n}}\kappa^{\frac{b}{m}}\ldots,$$

which would not involve the radical $z^{(k)}$; so that in this way the number of extractions of prime roots of variables might be diminished, which would be inconsistent with the irreducibility of $v^{(m)}$. (This is exemplified, by Sir W. R. Hamilton, in the case of a cubic.)

[10.] Since each of the $t^{(m)}$ terms of the development of $v^{(m)}$ can be expressed as a rational function of the s roots,

$x_1, \ldots x_s$, of that equation of the s^{th} degree which $v^{(m)}$ is supposed to satisfy; it follows that every rational function of these $t^{(m)}$ terms must likewise be a rational function of those s roots, and must admit, as such, of some finite number, r, of values, corresponding to all possible changes of arrangement of the same s roots among themselves. The same term or function must, for the same reason, be itself a root of an equation of the r^{th} degree, of which the coefficients are symmetrical functions of the s roots, $x_1, \ldots x_s$, and therefore are rational functions of the s coefficients, $A_1, \ldots A_s$, and ultimately of the n original quantities $a_1, \ldots a_n$; while the $r-1$ other roots of this new equation are the $r-1$ other values of the same function of $x_1, \ldots x_s$, corresponding to the changes of arrangement just now mentioned. Hence, every one of the $n^{(m)} + t^{(m)}$ functions $(T^{(m)})^{m'}$ and $U^{(m-1)}$, and therefore also every one of the $n^{(m)} + t^{(m)}$ functions f' and w', to which they are respectively equal, and which have been shown to contain, among them, all the radicals of orders lower than m, must be a root of some such new equation, although the degree r will not in general be the same for all. Treating these new equations and functions, and the radicals of the order $m-1$, as the equation $x^s + \&c. = 0$, the function $v^{(m)}$, and the radicals of the order m have been already treated; we obtain a new system of relations, analogous to those already found, and capable of being thus denoted:

$$T^{(m-1)} = C^{(m-2)} z^{(m-1)};$$

$$(T^{(m-1)})^{m''} = (C^{(m-2)})^{m''} f'';$$

$$U^{(m-2)} = u' \,(C_1^{(m-2)})^{-\alpha'}\, (C_2^{(m-2)})^{-\beta'} \ldots = v''.$$

And, so proceeding, we come at last to a system of the form

$$T' = Cz',$$

in which the coefficient C is different from zero, and is a rational function of the n original quantities $a_1, \ldots a_n$; while T is a rational function of the s roots, $x_1, \ldots x_s$, of that equation of the s^{th} degree in x which it has been supposed that $v^{(m)}$ satisfies. We have therefore the expression

$$z' = \frac{T'}{C},$$

which enables us to consider every radical z, of the first order, as a rational function F' of the s roots $x_1, \ldots x_s$, and

of the n original quantities $a_1, \dots a_n$: so that we may write for each radical, z', of the first order

$$z' = F'(x_1, \dots x_s, \ a_1, \dots a_n).$$

But before arriving at the last mentioned system of relations, another system of the second order and of the form

$$T'' = C'z''$$

must have been found, in which C' is different from zero, and is a rational function of z_1', z_2', &c., and of $a_1, \dots a_n$, while T'' is a rational function of $x_1, \dots x_s$; we have therefore the expression

$$z'' = \frac{C'}{T''},$$

and we see that every radical of the second order also is equal to a rational function of $x_1, \dots x_s$ and of $a_1, \dots a_n$: so that we may write

$$z'' = F''(x_1, \dots x_s, \ a_1, \dots a_n).$$

And, reascending thus, through orders higher and higher, we find, finally, by similar reasonings, that every one of the

$$n' + n'' + \dots + n^{(k)} + \dots + n^{(m)}$$

radicals which enter into the composition of the irrational and irreducible function $v^{(m)}$, such as the radical $z^{(k)}$, must be expressible as a rational function $F^{(k)}$ of the roots $x_1, \dots x_s$, and of the original quantities $a_1, \dots a_n$: so that we have a complete system of expressions, for all these radicals, which are included in the general formula

$$z^{(k)} = F^{(k)}(x_1, \dots x_s, \ a_1, \dots a_n).$$

Some illustration of this is afforded by a cubic.

[11.] In general, let p be the number of values which the rational function $F^{(k)}$ can receive, by altering in all possible ways the arrangement of the s roots $x_1, \dots x_s$, these roots being still treated as arbitrary and independent quantities, (so that p is equal either to the product $1.2.3\dots s$, or to some submultiple of that product); we shall then have an *identical* equation of the form

$$F^p + D_1 F^{p-1} + \dots + D_{p-1} F + D_p = 0,$$

in which the coefficients $D_1, \dots D_p$ are rational functions of $a_1, \dots a_n$; and therefore at least *one* value of the radical $z^{(k)}$ must satisfy the equation

$$z^p + D_1 z^{p-1} + \dots + D_{p-1} z + D_p = 0.$$

But in order to do this, it is necessary, for reasons already explained, that *all* the values of the same radical $z^{(\omega)}$, obtained by multiplying itself and all its subordinate radicals of the same functional system by any powers of the corresponding roots of unity, should satisfy the same equation; and therefore that the number q of these values of $z^{(\omega)}$ should *not exceed* the degree p of that equation, or the number of the values of the rational function $F^{(\omega)}$. Again, since we have denoted by q the number of values of the radical, we must suppose that it satisfies identically an equation of the form

$$z^q + E_1 z^{q-1} + \ldots + E_{q-1} z + E_q = 0,$$

the coefficients $E_1, \ldots E_q$ being rational functions of $a_1, \ldots a_n$; and therefore that at least one value of the function $F^{(\omega)}$ satisfies the equation

$$F^q + E_1 F^{q-1} + \ldots + E_{q-1} F + E_q = 0.$$

Suppose now that the s roots $x_1, \ldots x_s$ of the original equation in x,

$$x^s + A_1 x^{s-1} + \ldots + A_{s-1} x + A_s = 0,$$

are really unconnected by any relation among themselves, a supposition which requires that s should not be greater than n, since $A_1, \ldots A_s$ are rational functions of $a_1, \ldots a_n$; suppose also that $a_1, \ldots a_n$ can be expressed, reciprocally, as rational functions of $A_1, \ldots A_s$, a supposition which requires, reciprocally, that n should not be greater than s, because *the original quantities $a_1, \ldots a_n$ are, in this whole discussion, considered as independent of each other.* With these suppositions, which involve the equality $s = n$, we may consider the n quantities $a_1, \ldots a_n$, and therefore also the q coefficients $E_1, \ldots E_q$, as being symmetric functions of the n roots $x_1, \ldots x_n$ of the equation

$$x^n + A_1 x^{n-1} + \ldots + A_{n-1} x + A_n = 0;$$

we may also consider $F^{(\omega)}$ as being a rational but unsymmetric function of the same n arbitrary roots, so that we may write

$$z^{(\omega)} = F^{(\omega)}(x_1, \ldots x_n);$$

and, since the truth of the equation

$$F^q + E_1 F^{q-1} + \ldots + E_{q-1} F + E_q = 0$$

must depend only on the *forms of the functions*, and not on the *values of the quantities* which it involves (those values being altogether arbitrary), we may alter in any manner

the arrangement of these n arbitrary quantities $x_1, \ldots x_n$, and the equation must still hold good. But by such changes of arrangement, the symmetric coefficients $E_1, \ldots E_q$ remain unchanged, while the rational but unsymmetric function $F^{(k)}$ takes, in succession, all those p values of which it was before supposed to be capable; these p unequal values therefore must all be roots of the same equation of the q^{th} degree, and consequently q must *not be less* than p. And since it has been shown that the former of these two last mentioned numbers must *not exceed* the latter, it follows that they must be *equal* to each other, so that we have the relation

$$q = p:$$

that is, the radical $z^{(k)}$ and the rational function $F^{(k)}$ must be exactly *coextensive in multiplicity of value.*

This is exemplified by the cubic.

[12.] The conditions assumed in the last article are all fulfilled, when we suppose the coefficients A_1, &c. to coincide with the n original quantities a_1, &c., that is, when we return to the equation originally proposed;

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0,$$

which is the general equation of the n^{th} degree: so that we have for any radical $z^{(k)}$, which enters into the composition of an irrational and irreducible function representing any root of any such equation, an expression of the form

$$z^{(k)} = F^{(k)}(x_1, \ldots x_n);$$

the radical and the rational function being coextensive in multiplicity of value. We are, therefore, conducted thus to the important theorem, to which Abel was first led, by reasonings somewhat different from the foregoing. Examples of the truth of this theorem are given by anticipation in arts. [7.] and [10.] of Sir W. R. Hamilton's essay; and Sir W. R. Hamilton here adds an illustration derived from a biquadratic equation.

But before he proceeds to apply the theorem, in a manner similar to that of Abel, Sir W. R. Hamilton shows *a priori*, with the help of the same general theorem, that no new finite function, irrational and irreducible, can be found, essentially distinct in its radicals from those which have long since been discovered, for expressing any root of such lower but general equation, quadratic, cubic, or biquadratic, in terms of the coefficients of that equation.

[13.] Contains a discussion of the general quadratic, and [14.] of the general cubic.

[15.] Functions are defined as *syntypical* when they are formed from one common type. $F(x_1, x_2)$ and $F(x_2, x_1)$ are syntypical as formed from one common type $F(x_\alpha, x_\beta)$. It is shown that two syntypical functions of two arbitrary quantities cannot have equal cubes, if they be themselves unequal.

[16.] and [17.] It is shown that if an unsymmetric rational function of three arbitrary quantities have fewer than six values, it must be reducible either to the two-valued form

$$a + b\,(x_1 - x_2)\,(x_1 - x_3)\,(x_2 - x_3),$$

or to the three-valued form

$$a + bx + cx^2.$$

[18.] Functions of the form

$$b\,(x_\alpha - x_\beta)\,(x_\alpha - x_\gamma)\,(x_\beta - x_\gamma),$$

are the only two-valued functions of three variables which have symmetric squares; the square of a three-valued function of three variables is always itself three-valued; if an unsymmetric function of three variables have fewer than six values, its cube cannot have fewer values than itself; and if a rational function of three arbitrary quantities have a symmetric cube it must be itself symmetric. The form

$$\{a + b\,(x_\alpha - x_\beta)\,(x_\alpha - x_\gamma)\,(x_\beta - x_\gamma)\}\,(x_\alpha + \omega_3{}^2 x_\beta + \omega_3 x_\gamma)$$

of six-valued function of three variables has a two-valued cube.

[19.] A rational function of four symbols may have twenty-four, twelve, eight, six, four, or three values. Of twenty-four-valued functions two forms have twelve-valued squares, and one an eight-valued cube. Three forms of twelve-valued functions have six-valued squares. An eight-valued function may have its square four-valued. Three forms of six-valued functions have three-valued squares, and a modification of one of the latter forms has a two-valued cube; and the square of any multiple of the discriminant is symmetric. But there exists no other case of reduction essentially distinct from those discussed by Sir W. R. Hamilton, in which the number of values of

the square or cube of a rational function of four independent variables is less than the number of values of that function itself.

[20.] It is impossible to discover any essentially new expression for the roots of a biquadratic. And the only important difference with respect to the extraction of radicals between any two general methods for resolving biquadratics, if both be free from all superfluous extractions, is, that after calculating first, in both methods, a square root and a cube root, we may afterwards either calculate two *simultaneous* square roots, as in the method of Euler, or else two *successive* square roots, as in the method of Ferrari and Descartes:—for, in the view in which they are here considered, the methods of these two last-mentioned mathematicians do not essentially differ from each other.

[21.] It is not necessary, for the purposes of the inquiry into the possibility or impossibility of representing, by any expression of the form v, a root x of the general equation of the fifth degree,

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0,$$

to investigate all possible forms of rational functions of five variables, which have fewer than 120 values; but it is necessary to discover all those forms which have five or fewer values. These forms are investigated, and it is shown that there are none with fewer than six values except only

$$a + b \times \text{the discriminant},$$

and the five-valued function

$$b_0 + b_1x_a + b_2x_a^2 + b_3x_a^3 + b_4x_a^4;$$

the coefficients b_0, b_1, b_2, b_3, b_4 being symmetric.

It is then proved that a multiple of the discriminant is the only form which has a prime power (namely its square) symmetric; and that no form can have any prime power two-valued if its own values be more numerous than two.

[22.] Since the coefficients $a_1, \ldots a_5$ are symmetric functions of the roots $x_1, \ldots x_5$, it is clear that we cannot express any one of the latter as a rational function of the former; m in $v^{(m)}$, must therefore be greater than 0; and the expression $v^{(m)}$, if it exist at all, must involve at least one radical of the first order, z', which must admit of being expressed as a rational but unsymmetric function F' of the five roots,

but must have a prime power symmetric, and consequently must be a square root and therefore a multiple of the discriminant. And because any other radical of the same order can be deduced from this by multiplication, we see that no other radical of the first order can enter into $v^{(m)}$ when that expression is cleared of all superfluous functional radicals. On the other hand, a two-valued expression cannot represent the five-valued function x; if then the sought expression $x=v^{(m)}$ exist at all, it must involve some radical of the second order, z'', and this radical must admit of being expressed as a function F'' of the five roots, which function is to have, itself, more than two values, but to have some prime power two-valued. And since it has been proved that no such function F'' exists, it follows that no function of the form $v^{(m)}$ can represent the sought root x of the general equation of the fifth degree.

[23.] *A fortiori* it is impossible to satisfy the general equation $x^n+\&c.=0$ when n is greater than five.

[24.] Contains the criticism on Abel which I have already adverted to, and the remark that another chief obscurity in Abel's argument had rendered it advantageous, as preliminary to the discussion of the forms of functions of five arbitrary quantities, to establish certain auxiliary theorems respecting functions of fewer variables; which have served also to determine *a priori* all possible solutions by radicals and rational functions (for Sir W. R. Hamilton does not absolutely ignore others, see the end of [22.] p. 247 and of [28.] p. 255) of all general algebraic equations below the fifth degree.

[25.] Contains Abel's proof, after Cauchy, that if a function of five variables have fewer than five values, it must be either two-valued or symmetric.

[26.], [27.], and [28.] Contain remarks on Mr. Jerrard's method and a consideration of his trinomial form of quintic. Sir W. R. Hamilton proposes a mode of solving it; not, of course, algebraically, but by *two new tables of double entry.*

In an "Addition" (pp. 256—259) Sir W. R. Hamilton shows that the elegant analysis of Mr. Murphy fails to establish any conclusion opposed to the argument of Abel.

4, Pump Court, Temple, London,
***September* 3, 1860.**

ON THE COVARIANTS OF A BINARY QUANTIC OF THE n^{th} DEGREE.

By MICHAEL ROBERTS.

I PROPOSE, in what follows, to continue the discussion which has been the subject of my former papers. The method which I have employed in them is, I think, peculiarly applicable to the discovery of relations existing between functions of the differences of the roots of equations of the higher degrees. These functions, when considered by themselves as a set of individual terms, are generally of such repulsive length and apparent complexity as to afford no clue whatever to the equations by which they are connected with other functions of the same class, and as they are the analytical elements on which the nature of the roots essentially depends, it is plain that any inquiry into their properties must be valuable when considered in reference to the progress of algebraic science. A remarkable equation of M. Brioschi, which I have already had occasion to quote, serves to connect the theory of functions of the differences of the roots of algebraic equations with the theory of the covariants of binary quantics: indeed, from my point of view, after this *rapprochement* has been made, the two theories become identical. As this paper is to be read in connexion with my Memoir (*Quarterly Journal*, Vol. IV., pp. 168–178), the notation is the same in both.

Suppose now that x_r is a root of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0,$$

if we put $x_r = -\dfrac{a_1}{a_0} + u$, and if we operate with the symbol δ of my former paper, we find

$$\delta x_r = -1 + \delta u;$$

but $\delta x_r = -1$ so that $\delta u = 0$, or u (a function capable of supplying all the roots) is a function of the differences of the roots. One of the most remarkable peculiarities of these functions is the great facility with which they give rise to other functions of the same kind, and the flexibility with which the relations existing between them admit of being

transformed. For example, let there be m covariants of the quantic $(a_0, a_1, a_2, \ldots a_n)(x, y)^n$, namely,

$$(R_0, R_1, \ldots R_p)(x, y)^p,$$
$$(S_0, S_1, \ldots S_q)(x, y)^q,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(T_0, T_1, \ldots T_r)(x, y)^r,$$

then the determinant

$$\begin{vmatrix} R_0, R_1, \ldots R_{m-1} \\ S_0, S_1, \ldots S_{m-1} \\ \ldots\ldots\ldots\ldots\ldots \\ T_0, T_1, \ldots T_{m-1} \end{vmatrix}$$

is a function of the differences of the roots of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0,$$

and therefore the source of a covariant of the original quantic.

If θ is the quadrinvariant of the form $(a_0, a_1, \ldots a_{2k})(x, y)^{2k}$ it is the source of a covariant of the form $(a_0, a_1, \ldots a_{2k+1})(x, y)^{2k+1}$ of the second degree in the variables: let us denote it by

$$\theta x^2 + \theta_1 xy + \theta_2 y^2,$$

and if ψ is the quartinvariant of the form

$$(a_0, a_1, a_2, \ldots a_{2k+1})(x, y)^{2k+1},$$
$$\psi = 4\theta\theta_2 - \theta_1^2,$$

and
$$\tfrac{1}{2}\frac{d\psi}{da_{2k+1}} = 2a_1\theta - a_0\theta_1,$$

hence
$$a_0^2\psi + \tfrac{1}{4}\frac{d\psi^2}{da^2_{2k+1}}$$

has θ for a factor, and it is easy to see that this expression does not contain a_{2k+1}.

The remainder of this paper will be exclusively occupied with an examination of the covariants of binary quintics, and, when taken in connexion with my former papers, will, I think, furnish a tolerably complete theory of these functions.

I have already shewn that all covariants of binary quintics, whose source is an even function of the roots, can be expressed by the quintic itself, the covariants (H), (I), (J), (L), and the invariant K: these quantities being connected by the covariant equation derived from the relation

$$a_0^2L^2 = 4H\{H(I^3 - 9J^2 - 2IL - HK) + a_0J(3L - I^2)\}$$
$$+ a_0^2K(HI + a_0J) + a_0^2I(12J^2 + 2IL - I^3) \ldots\ldots\ldots(a).$$

If the source of the covariant is an odd function of the roots, the covariants (G), (K') must be combined with the above-mentioned functions, and between all these covariants there exist three syzygies, namely, the covariant equations which flow from the following relations:

$$G^2 = 4H^3 - a_0^2 HI - a_0^3 J, \quad K'^2 = 4HI^2 - 12a_0 IJ - a_0^2 K,$$
$$GK' = 4H^2 I - 6a_0 HJ + a_0^2 (L - I^2) \,\ldots\ldots\ldots (b),$$

(*Quarterly Journal*, Vol. IV., p. 173). The equation (a) is obtained by equating the two values of $G^2K'^2$ derived from the group (b).

Hence the covariant equation of the equation of the squares of the differences of a quintic depends on the quintic itself, the covariants (H), (I), (J), (L), and the invariant K: it is at once deduced from Mr. Cayley's equation (given in the *Philosophical Transactions* for 1860) by putting the second term of the quintic equal to zero, and by substituting for the remaining terms the values given in my paper (*Quarterly Journal*, Vol. IV., p. 172).

If ϕ is the coefficient (to a factor *près*) of the penultimate term of the equation of the squares of the differences of a quintic (as given in the *Quarterly Journal*, Vol. IV., p. 327), the covariant whose source is ϕ is of the fourth degree in the variables, and furnishes values for a double root by the successive quotients of its coefficients: its quadrinvariant has for a factor the discriminant of the quintic. (This result is general, and applies to equations of higher degrees). In consequence of the values which I have given for ϕ (*Quarterly Journal*, Vol. IV., p. 234) and for the third Sturmian constant S_3 for a quintic (*Quarterly Journal*, Vol. IV., p. 175), I find

$$a_0\phi + 2JS_3 = K\{50HJ + a_0(3I^3 + L)\}$$
$$- 2I\{a_0 IK - 12J(3L - I^3) + 8HR\},$$

an expression which must vanish for a triple root and also for a pair of double roots. Now Mr. Cayley has shewn (*Philosophical Transactions*, Vol. CXLVII., p. 730) that for a pair of double roots,

$$50HJ + a_0(3I^3 + L) = 0,$$

so that the function

$$a_0 IK - 12J(3L - I^3) + 8HR$$

vanishes for a triple root, and also for a pair of double roots.

At the conclusion of my paper, Vol. IV., p. 327, I derived

a remarkable function of the differences of the roots of a quintic from similar functions of a sextic. This function which I have called T admits of an elegant mode of derivation from the quintic itself. Suppose that the cubicovariant (J) is represented by

$$(J, J_1, J_2, J_3)(x, y)^3,$$

I find
$$T = 9(J_1^2 - JJ_2):$$

in fact the covariant (T), which for a quintic is of the second degree in the variables, is the Hessian of (J).

The equation (a) by the employment of T' can be written

$$L^2 = 4HT + K(HI + a_0J) + I(12J^2 + 2IL - I^3).$$

If $(L, L_1, L_2, L_3, L_4)(x, y)^4$ is the covariant whose source is L, then

$$L_1^2 - LL_2 = IT + JR,$$

so that the Hessian of (L) is $(I)(T) + (J)(R)$.

If Ω denotes the octinvariant of the quintic (No. 25 of Mr. Cayley's Tables) I find the quadrinvariant of $h(L) + k(I)^2$ to be $h(h+k)\Omega + k^2K^2$, so that the quadrinvariant of No. 20 of Mr. Cayley's Tables is K^2.

I find also

$$a_0\Omega = J(12T - IK) + R(L + I^2) \ldots\ldots\ldots\ldots (c).$$

By means of this expression and those which I have already given, the Sturmian constants for a quintic are expressed in a form which renders their calculation sufficiently simple, as after having calculated the values of H, I, J, K, L, which are not very complicated, those of R and T can be readily ascertained from the following equations, which I have given before,

$$a_0R = I^3 - 9J^2 - 2IL - HK \ldots\ldots\ldots\ldots\ldots (d),$$
$$a_0T = RH + (3L - I^2)J \ldots\ldots\ldots\ldots\ldots\ldots (e).$$

We have then the following table of the Sturmian constants for a quintic,

$$S_1 = H,\ S_2 = 5HI + 9a_0J,\ S_3 = HK + 12IL + 4I^3 - 216J^2,$$
$$a_0S_4 = a_0K^2 - 128\{J(12T - IK) + R(L + I^2)\}.$$

In an interesting paper by Mr. Cayley, which has appeared in this *Journal* (Vol. IV., pp. 7–12), the criteria for the number of real roots in cubic and quartic equations are deduced from an examination of the Sturmian constants of

these equations. The author has also verified by *à posteriori* reasoning, the cases relative to the signs of the Sturmian constants, which it is clear, *à priori*, are impossible. I propose to demonstrate, in a similar manner, that the following consecution of signs, − + −, for the first three Sturmian constants is impossible for a quintic. Let

$$\Sigma = \tfrac{1}{2}\frac{dS_3}{da_5},$$

I find then
$$\Sigma = HK' - 6IG,$$

whence $\Sigma^2 = -a_0^2HS_3 - 4S_2(a_0^2I^2 + 6a_0HJ - 5H^2I)$,

but the condition $S_2 > 0$ can be written in the form

$$-4H(a_0^2I - 9H^2) - 9G^2 > 0,$$

and as S_1 or H is negative, it follows that

$$a_0^2I - 9H^2 > 0:$$

and

$$a_0^2(a_0^2I^2 + 6a_0HJ - 5H^2I) = (a_0^2I - 3H^2)(a_0^2I - 8H^2) - 6HG^2,$$

but from the condition which I have just obtained, it follows that $a_0^2I - 3H^2$, and $a_0^2I - 8H^2$ are both positive, so that

$$a_0^2I - 6a_0HJ - 5H^2I > 0.$$

Hence from the value found for Σ^2, it is plain that S_2 cannot be negative.

The following equation is worth noticing,

$$JS_3 + 4RS_2 = 5\{5HJK + 12a_0RJ + 4a_0IT\}.$$

I find also,

$$\tfrac{1}{4}\frac{dR^2}{da_5^2} = a_0\{RH - I^2J\} + 9HJ^2:$$

this equation proves that when H is negative, R and J cannot both be positive.

We have also

$$\left(\frac{dI}{da_5}\right)^2 = 4H^2I + 3J^2(HI - 3a_0J).$$

If C is the invariant of the twelfth degree of the quintic, the eliminant of (R) and (T) is

$$\frac{\Omega^2 - 3CK}{4},$$

and the eliminant of (R) and (I) is $K\Omega - 9C$: these results are deduced by employing Mr. Sylvester's canonical form, which is admirably adapted for expressing composite invariants, such as the eliminants of covariants, in terms of the fundamental invariants. The eliminant then of (R) and $12(T) - K(I)$ is

$$\Omega(3\Omega - K^2):$$

and the eliminant of (R) and (J) is, as I have already shewn, (*Quarterly Journal*, Vol. IV., p. 327) the skew-invariant of M. Hermite, which I shall denote by Γ. Hence we infer from the covariant equation of (c) that the eliminant of the quintic and its linear covariant is

$$\Gamma(3\Omega - K^2).$$

(This result can also be obtained by seeking directly the eliminant of the quintic and its linear covariant). If

$$Rx + R_1 y$$

represents this latter covariant, then the quantity $-\dfrac{R_1}{R}$ is a root of the equation

$$(a_0, a_1, a_2, a_3, a_4, a_5)(x, 1)^5 = 0,$$

if $\Gamma = 0$, or if $3\Omega - K^2 = 0$.

It is not difficult in general to express the eliminant of the linear covariant and any other covariant in terms of the fundamental invariants. For example, from the covariant equation of (e) we find the eliminant of (R) and (L) to be

$$\tfrac{1}{4}\{\Omega^3 - 27K\Omega C + K^2\Omega^2 + K^3C + 108C^2\},$$

and from this result and the covariant equation of (d) we find for the eliminant of (R) and (H)

$$\tfrac{1}{16}\{8K^2\Omega^3 - 126K\Omega^2C + 432\Omega C^2 + 81K^2C^2 + \Omega^4 - 8K^3\Omega C\}.$$

It may easily be shewn that the eliminant of (R) and the covariant (W) which I have already noticed (*Quarterly Journal*, Vol. IV., p. 325) is identical with the eliminant of (R) and the quintic itself. For putting

$$U = (a_0, a_1, a_2, a_3, a_4, a_5)(x, y)^5,$$

we have

$$5U = x\frac{dU}{dx} + y\frac{dU}{dy}, \quad 5(W) = R\frac{dU}{dy} - R_1\frac{dU}{dx},$$

and from these equations, the proposition which I have just enunciated follows at once. The eliminants of (R) and (G)

and of (R) and (K') can therefore be derived from the values which I have given for WG, WK' (*Quarterly Journal*, Vol. IV., p. 325) and it is also plain that the eliminant of U and (W) is

$$\Gamma(3\Omega - K'^2) \times \text{discriminant of } U.$$

We may remark also that Γ is the eliminant of U and (J). I have already remarked that for a triple root the equations

$$K = 0, \quad 3L - I^2 = 0$$

coexist: it is then an interesting inquiry to ascertain whether their coexistence necessarily implies the existence of a triple root. If the above-mentioned conditions hold, we find from equation (a)

$$3a_0^2L^2 = 4H^2(I^3 - 27J^2) + a_0^2I(36J^2 - I^3),$$

and therefore in consequence of the given conditions

$$27a_0^2L^3 = 12H^2I^3(I^3 - 27J^2) + 3a_0^2I^3(36J^2 - I^3).$$

Hence

$$a_0^2I^4 = 12H^2(I^3 - 27J^2) + 3a_0^2I(36J^2 - I^3),$$

which gives

$$(I^3 - 27J^2)(a_0^2I - 3H^2) = 0,$$

so that the coexistence of the equations $K = 0$, $3L - I^2 = 0$ implies the existence of one or other of the equations

$$I^3 - 27J^2 = 0, \quad a_0^2I - 3H^2 = 0.$$

The first of these corresponds to the case of a triple root, the second indicates that the same transformation removes the coefficients a_1, a_4 from the equation

$$(a_0, a_1, a_2, a_3, a_4, a_5)(x, 1)^5 = 0,$$

and in this case the roots are connected by the relations

$$\Sigma(x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_4)^2(x_4 - x_5)^2(x_5 - x_1)^2 = 0,$$

$$\Sigma\frac{1}{4x_1 - x_2 - x_3 - x_4 - x_5} = 0.$$

POSTSCRIPT.—It may easily be shewn that with the consecution of signs $- + +$ for the first three Sturmian constants of a quintic, S_4 or the discriminant is positive.

By putting the coefficient of the second term of a quartic equal to nothing in the expression for its discriminant and by

making the substitutions which I have already so frequently employed, the following identical relation is easily established:

$$a_0^6 (I^3 - 27J^2) = (a_0^2 I - 3H^2)(a_0^2 I - 12H^2)^2$$
$$+ 27 G^2 \{a_0^3 J - H(a_0^2 I - 4H^2)\}.$$

From what I have already stated, it is plain that when H is negative and $5HI + 9a_0 J$ (or the second Sturmian constant for a quintic) is positive, I, J, and $a_0^2 I - 4H^2$ are positive: and we infer from the formula which I have just given that in this case $I^3 - 27J^2$ is positive: and the four roots of the equation

$$(a_0, a_1, a_2, a_3, a_4)(x, 1)^4 = 0$$

are imaginary. Let us denote them by α, β, γ, δ; and as this equation is the derived of $(a_0, a_1, a_2, a_3, a_4, a_5)(x, 1)^5 = 0$, the discriminant of this latter is equal to

$$a_0^4 \times \begin{cases} (a_5 + 5a_4\alpha + 10a_3\alpha^2 + 10a_2\alpha^3 + 5a_1\alpha^4 + a_0\alpha^5) \\ \times (a_5 + 5a_4\beta + 10a_3\beta^2 + 10a_2\beta^3 + 5a_1\beta^4 + a_0\beta^5) \\ \times (a_5 + 5a_4\gamma + 10a_3\gamma^2 + 10a_2\gamma^3 + 5a_1\gamma^4 + a_0\gamma^5) \\ \times (a_5 + 5a_4\delta + 10a_3\delta^2 + 10a_2\delta^3 + 5a_1\delta^4 + a_0\delta^5) \end{cases}$$

which is therefore essentially positive.

The formula which has been given for the discriminant of a quartic includes what has been demonstrated by Mr. Cayley (*Quarterly Journal*, Vol. IV., pp. 9, 10); namely, that for the sequence of signs $- +$ for the first two Sturmian constants for a quartic the discriminant is necessarily positive. In this case the condition $2HI + 3a_0 J > 0$ gives

$$-H(a_0^2 I - 12H^2) - 3G^2 > 0,$$

and as H is negative, $a_0^2 I - 12H^2$ is positive, and therefore $a_0^2 I - 4H^2$ is positive, from which we infer by means of our formula that $I^3 - 27J^2$ is positive.*

* The skew-invariant of the ninth degree which I mentioned as belonging to a sextic (*Mathematical Journal*, Vol. IV., p. 326) I find on closer examination to have no real existence, as indeed is evident from Mr. Cayley's theory. Mr. Cayley also informs me that the decadic has several invariants of the ninth degree, so that those invariants which I supposed to be identical are in reality different.

15, Trinity College, Dublin,
September 4, 1861.

ON THE EQUATION OF THE EVOLUTE OF A PARABOLA.

By J. Corbett Turnbull, M.A., Trinity College, Cambridge, and Head Mathematical Master of Cheltenham College.

By a well known property, if three normals be drawn from a point in this the parabola to the points (x_1y_1), (x_2y_2), (x_3y_3), we have

$$y_1+y_2+y_3=0,$$

which gives $y_1+y_2=y_3$ geometrically.

Let (x_2y_2) approach indefinitely near to (x_1y_1).

Then the point of intersection of these two normals is on the evolute, and $2y_1=y_3$.

And the locus of the intersection of the normals at (x_1y_1) and (x_3y_3) will be the evolute.

To find its equation, we have the equations to the two normals

$$y-y_1=-\frac{xy_1}{2a}+\frac{y_1^3}{8a^2},$$

$$y+2y_1=\frac{xy_1}{a}-\frac{y_1^3}{a^2},$$

subtracting and dividing by $3y_1$

$$1=\frac{x}{2a}-\frac{3y_1^2}{8a^2};$$

therefore $$3y_1^2=4a(x-2a);$$

therefore $$y^2=y_1^2\left\{1-\frac{x}{2a}+\frac{y_1^2}{8a^2}\right\}^2=\frac{4a(x-2a)}{3}\left\{\frac{6a-3x+x-2a}{6a}\right\}^2$$

$$=\frac{4a(x-2a)^3}{27a^2};$$

therefore $$27ay^2=4(x-2a)^3$$

is the equation of the evolute.

NOTE ON DES CARTES' RULE OF SIGNS.

By William Walton, M.A., Trinity College.

The object of this note is to give a demonstration of Des Cartes' Rule of Signs, not essentially different from that ordinarily given in systematic treatises on the Theory of Equations, but more lucid, I think, in the manner of presentation and therefore better adapted for the ready apprehension of students.

Let $U=0$, $V=0$, be two equations of the forms which are discussed in the Theory of Equations, and let the latter equation contain all the roots of the former and only one more, and that a positive one. Suppose the polynomials represented by U, V, to be arranged by ascending powers of x, and their first terms to be both positive. Let $u_r x^r$, $u_s x^s$, be two successive terms in U, and v_s the coefficient of x^s in V. I shall first prove the two following lemmas:

(A) If, s being equal to $r+1$, u_r and u_s have different signs, v_s has the same sign as u_s.

(B) If s be greater than $r+1$, v_s has the same sign as u_s.

Let c be the additional root of the equation $V=0$: then, since V is equal to $(c-x)U$, it is obvious that, t being any integer,

$$v_{t+1} = cu_{t+1} - u_t \text{ ....} (1).$$

The equation (1) renders evident the truth of the lemma (A).

Next conceive u_r and $u_{r+\mu+1}$ to be the coefficients of two successive terms in U, and accordingly u_t to be zero for all values of t between r and $r+\mu+1$: then, by the equation (1), it is evident that $v_{r+\mu+1}$ is equal to $cu_{r+\mu+1}$ or that $v_{r+\mu+1}$ has the same sign as $u_{r+\mu+1}$: that is, the lemma (B) is established.

From the lemmas (A) and (B) it is evident that, the upper of the two following rows of signs representing the signs of the successive groups of terms in the whole polynomial U, the lower will represent the signs of a corresponding series of groups of terms in V, the signs of the lower row being less numerous by one than those of the upper:

$$+,\ -,\ +,\ -,\ \ldots\ldots \pm$$
$$-,\ +,\ -,\ \ldots\ldots \pm.$$

But the first group of terms in V is positive, and, by the relation $V=(c-x)U$, the sign of the last must be $\mp$: these considerations present to our notice two additional changes of sign in V: thus the changes of sign in V must in all certainly exceed those in U by one. Thus the introduction of a new positive root must augment the number of changes of sign by at least one; and, accordingly, no equation, complete or incomplete, can have more positive roots than it has changes of sign; a proposition which constitutes Des Cartes' Rule of Signs in relation to the positive roots of an equation. His Rule in regard to the limit to the number of the negative roots follows at once as a corollary from his Rule for the positive roots, as is shewn in ordinary Treatises on the Theory of Equations.

July 26, 1861.

ON PETZVAL'S ASYMPTOTIC METHOD OF SOLVING DIFFERENTIAL EQUATIONS.

By William Spottiswoode, M.A., F.R.S., &c.

The researches of M. Petzval here brought under notice are directed to the solution of those Linear Differential Equations with variable coefficients which, to use his own words, have reference to small motions, and in which the dependent variable represents infinitesimal disturbances. He then points out that as these small motions not only subsist in the neighbourhood of their origin, but are propagated to finite distances, the solutions of the differential equations which correspond to large values of the independent variable become a legitimate and useful object of study. These solutions are however, as will be at once apprehended, not rigorous for the entire range of the phenomena, but exhibit only the leading terms of the integral expressions; in fact, those terms whose magnitude becomes preponderant when the value of the independent variable is indefinitely increased. In other words, if the complete expression for the dependent variable y, in terms of the independent x, be regarded as the equation to a curve, the expressions obtained by M. Petzval would represent those asymptotes which are situated in the neighbourhood of the axis of x. The method is, on this account, called the "Asymptotic Method of Solution."

A large part of his work, under the name of "Formenlehre," is devoted to the discussion of the forms of differential equations constructed from given particular integrals. And with a view to the conversion of his propositions, *i.e.*, to drawing conclusions with respect to integrals from the form of the differential equations, converse to those which he establishes with respect to the form of the equations from given integrals, he attaches great weight to a classification of functions given below.

To the *first class* belong such algebraic functions as have the following properties:

(1) That for increasing values of the independent variable x, their value approximates to that of a certain power of x, x^p, where p may be positive, negative, or fractional.

(2) That their differential coefficients, for infinite values of the independent variable x, are successively lower by unity in dimensions; and for those finite values of x which make them infinite, they are successively higher by unity in dimensions.

To the *second class* belong all functions possessing the same properties with respect to their differential coefficients as functions of the form $\varepsilon^{\int \phi(x)dx}$, where $\phi(x)$ is a function of the first class.

A third class might be formed by conceiving $\phi(x)$ itself to be of the second class; and so on indefinitely.

The coefficients of all differential equations here treated of are supposed to be of the first class.

He next proceeds to consider the effect produced on a given equation

$$X_n y^{(n)} + X_{n-1} y^{(n-1)} + \ldots + X_1 y' + X_0 y = 0 \;\ldots\ldots\ (1),$$

(where $y^{(i)}$ is the i^{th} differential coefficient of y with respect to x) by introducing a new particular integral, y_1, into the expression for y; the general integral of (1) containing n arbitrary constants.

Let z be the general integral of the new differential equation to be so formed; then

$$z = y + C_1 y_1$$

or

$$\left.\begin{aligned} y &= z - C_1 y_1 \\ y' &= z' - C_1 y_1' \\ &\cdots\cdots\cdots\cdots \\ y^{(n)} &= z^{(n)} - C_1 y_1^{(n)} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ (2),$$

whence
$$\left.\begin{aligned} &X_n z^{(n)} + X_{n-1} z^{(n-1)} + \dots X_1 z' + X_0 z \\ &= C_1 \{X_n y_1^{(n)} + X_{n-1} y_1^{(n-1)} + \dots X_1 y_1' + X_0 y\}\end{aligned}\right\} \dots\dots(3),$$

or writing
$$\left.\begin{aligned} X_n z^{(n)} + X_{n-1} z^{(n-1)} + \dots X_1 z' + X_0 z &= P \\ X_n y_1^{(n)} + X_{n-1} y_1^{(n-1)} + \dots X_1 y_1' + X_0 y_1 &= P_1\end{aligned}\right\} \dots\dots (4),$$

the equation (3) takes the form

$$P = C_1 P_1 \dots\dots\dots\dots\dots\dots\dots\dots\dots(5),$$

whence
$$P' : P = P_1' : P_1 = M : N \dots\dots\dots\dots\dots(6),$$

where M and N are polynomials cleared of fractions.

From (6) we may deduce

$$\begin{vmatrix} P, & P' \\ P_1, & P_1' \end{vmatrix} = 0, \quad \text{or} \begin{vmatrix} P, & P' \\ M, & N \end{vmatrix} = 0 \dots\dots\dots\dots(7),$$

the latter of which is the new differential equation in z of the order $(n+1)$, from which the arbitrary constant C_1 as well as the exponential factor belonging to the new particular integral y_1, have been eliminated.

If there be more than one particular integral introduced at once, *e.g.*, r of them, we may deduce the expressions corresponding to (7) as follows: the expression for z will be

$$z = y + C_1 y_1 + C_2 y_2 + \dots C_r y_r \dots\dots\dots\dots\dots(8),$$

or
$$y = z - C_1 y_1 - C_2 y_2 - \dots C_r y_r \dots\dots\dots\dots\dots(9),$$

with its successive differential coefficients up to the order z. Then, if $P_2, P_3, \dots P_r$, be the same functions of $y_2, y_3, \dots y_r$, that P_1 is of y_1, we shall have, by successive differentiation,

$$\left.\begin{aligned} P &= C_1 P_1 + C_2 P_2 + \dots C_r P_r \\ P' &= C_1 P_1' + C_2 P_2' + \dots C_r P_r' \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P^{(r)} &= C_1 P_1^{(r)} + C_2 P_2^{(r)} + \dots C_r P_r^{(r)}\end{aligned}\right\} \dots\dots\dots(10),$$

whence, by elimination,

$$\begin{vmatrix} P, & P_1, & \dots P_r \\ P', & P_1', & \dots P_r' \\ \dots & \dots & \dots \\ P^{(r)}, & P_1^{(r)}, & \dots P_r^{(r)} \end{vmatrix} = 0 \dots\dots\dots\dots\dots(11),$$

in which, as in (7), $P_1, P_1', \dots P_2, P_2', \dots$ may be replaced by quantities analogous to M, N.

Returning to (7), the developed expression is

$$\begin{aligned} z^{(n+1)}NX_n + z^{(n)} &\{NX_n' + NX_{n-1} - MX_n\} \\ + z^{(n-1)} &\{NX'_{n-1} + NX_{n-2} - MX_{n-1}\} \\ + &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + z' \quad &\{NX_1' + NX_0 - MX\} \\ + z \quad &\{NX_0' \qquad\qquad - MX_0\} = 0 \ldots (12). \end{aligned}$$

The author then proceeds to discuss, in detail, the characteristics, deducible from equation (12), of (1) entire algebraic functions, and others connected with them; (2) of exponential functions; (3) fractional functions.

And first, as regards *entire algebraic functions*. Suppose that

$$y_1 = Q,$$

where Q is a finite polynomial in x arranged in powers of that variable. Then the degree of M is lower than that of N by unity.

Pour fixer les idées, let all the X's be of the same order p; and let q be the order of N; then the coefficients of the z's will be all of the order $p+q$, excepting the last, which is of the order $p+q-1$. The same will be the case even if Q is fractional; for then P_1' takes the form $R : S$; and

$$\frac{P_1'}{P_1} = \frac{R'}{R} - \frac{S'}{S}, \quad \text{or } \frac{M}{N} = \frac{R'S - RS'}{RS} \ldots\ldots\ldots (13),$$

so that the degree of M is again lower by unity than that of N. In the cases then that we have considered, the introduction of a new particular integral results in the depression of the order of the last coefficient by unity, as compared with that of the last but one.

A depression will still take place, as may be seen on inspection, even if the order of X_0, instead of being equal to that of X_1, should exceed it by any dimensions.

If the order of X_0 be less than that of X_1, then the introduction of a new particular integral will cause the orders of the last *three* coefficients in the transformed equation to form a descending series.

We cannot however simply convert these propositions and from the orders of the last $(n+1)$ coefficients of a differential equation conclude the existence of r particular integrals of an algebraic form; because, as is hereafter shown, the same results as deduced above are produced by the introduction

of particular integrals of the form $y_1 = \varepsilon^{\psi(x)} Q$, where $\psi(x)$ is an algebraic fraction. These are, however, distinguishing criteria, in the case of fractions, considered hereafter.

We now come to the most important class for the present purpose, viz., exponential functions; and investigate the effect upon a given differential equation produced by the introduction of a new particular integral of that form. Beginning with the simplest case

$$y_1 = \varepsilon^{ax} Q \text{ ....} (14),$$

where Q is an entire algebraic polynomial, $y^{(r)}$ is of the form $\varepsilon^{ax} Q_r$, and consequently,

$$P_1 = \varepsilon^{ax} R,$$

whence
$$\frac{M}{N} = \frac{aR + R'}{R} \text{ ....} (15),$$

so that M and N are of the *same degree*.

Referring again to the test, of equation (12), we find that,

(1) If among the degrees of the coefficients of the given equation (1) there is a level (*i.e.*, two consecutive coefficients of the same degree); the same will be the case in the transformed equation (12). In special cases there may be even a fall.

(2) If there be a gradual fall in the degrees of the last k coefficients, and a level among the r preceding, in (1); there will, in (12), be likewise a gradual fall in the last k coefficients; but the level will extend over the $r + 1$ preceding.

The last may be further stated thus:

If the degrees of the X's rise from X_n to X_{n-k}, are thence level as far as X_k, and finally fall gradually to X_0; then, calling the transformed equation

$$Z_{n+1} z^{(n+1)} + Z_n z^{(n)} + \dots Z_1 z' + Z_0 z = 0 \text{ ....} (16),$$

the Z's will rise from Z_{n+1} to Z_{n+1-k}, remain level to Z_k, and then fall gradually to Z_0; the values of Z_{n+1-k}, and Z_k being

$$\left.\begin{aligned} Z_{n+1-k} &= NX'_{n+1-k} + NX_{n-k} - MX_{n+1-k} \\ Z_k &= NX'_k + NX_{k-1} - MX_k \end{aligned}\right\} \text{ ....} (17).$$

These indications are also common to the case where Q is fractional and of the first degree.

Next let us consider a particular integral of the form

$$y_1 = \varepsilon^{ax^2 + \beta x} Q, \quad \text{or } y_1 = \varepsilon^{\int (ax + \beta) dx} Q.$$

Then
$$P_1 = \varepsilon^{\int(\alpha x+\beta)dx} R,$$
and
$$\frac{M}{N} = \frac{(\alpha x+\beta)R + R'}{R},$$

and consequently the degree of M is *greater by unity* than that of N. And referring to the test equation (12), we conclude that:

(1) If the degrees of the X's are on a level, those of the Z's will rise from Z_{n+1} to Z_n, and thence remain on a level.

(2) Any continuous fall among the last k, X's will subsist also among the corresponding Z's.

(3) To a continuous rise among the first r coefficients of (1) there will, in general, correspond one among the $(r+1)$ first coefficients of (12). But between the first and last of this rise a fall among the Z's may occur.

The same properties belong also to all functions of the form $\varepsilon^{\psi(x)}Q$, the orders of $\psi(x)$, whose differential coefficients form an arithmetical series with a common difference unity.

More generally, let us consider the form
$$y_1 = \varepsilon^{\int\phi(x)dx} Q,$$
where $\phi(x)$ and Q are algebraic. Then
$$P_1 = \varepsilon^{\int\phi(x)dx} R,$$
and
$$\frac{M}{N} = \frac{\phi(x)R + R'}{R},$$

whence, the degree of M is greater than that of N, by that of $\phi(x)$.

After what has been said in the former cases it is necessary to particularize now only the case of the degree of $\phi(x)$ being fractional.

(A) Let the degree of $\int\phi(x)\,dx$ be $\frac{p}{q}$, a proper positive fraction. Then that of $\phi(x)$ is $\frac{p-q}{q} = -\frac{\delta}{q}$, where δ is positive; and the degree of M is less than that of N by $\frac{\delta}{q}$. Hence

(1) To a level among the orders of the last r coefficients of (1) corresponds a fall from the penult to the ultimate of (12), to the extent of $\frac{\delta}{q}$.

(2) The introduction of r such particular integrals will produce a fall among the last r coefficients of (12) amounting in all to $\frac{r\delta}{q}$.

(B) Let $\frac{p}{q}$ be an improper positive fraction. Then the degree of M is greater than that of N by $\frac{\delta}{q}$. Hence

(1) The introduction of a single particular integral of this form into an equation of which the initial X's are at a level will produce a rise between the first and second Z's to the extent of $\frac{\delta}{q}$.

(2) The introduction of r such particular integrals will produce a gradual rise among the first r Z's, each step being to the extent of $\frac{\delta}{q}$.

The author next considers the case of the presence of several particular integrals of the form $\varepsilon^{\int\phi(x)dx}Q$, where the degree of $\phi(x)$ is different for the different integrals.

Let us suppose first that the equation (1), into which two particular integrals of the present form are about to be introduced, has all its X's of the same degree p. Let μ, ν, σ, be the degree of M, N, $\phi(x)$; then

$$\mu = \nu + \sigma.$$

Also let A and B represent the degrees of the last two terms respectively of the trinomial coefficients of the z's in (12); then we have the following values for A, B, $B-A$,

A,	B,	$B-A$,	
$\nu+p$,	...........,	$-(\nu+p)$,	in the coefficient of $z^{(n+1)}$,
$\nu+p$,	$\nu+p+\sigma$,	σ,	.... $z^{(n)}$,
$\nu+p$,	$\nu+p+\sigma$,	σ,	.... $z^{(n-1)}$,
......,	...........,	...........,	...............................,

and the degrees of the coefficients (z's) of (12) will be

$$\nu+p,$$
$$\nu+p+\sigma,$$
$$\nu+p+\sigma,$$
$$\cdots\cdots\cdots$$

Introducing a second integral, for which the values of M, N, $\phi(x)$ are μ_1, ν_1, σ_1, we have

A,	B,	$B-A$,
$\nu_1+\nu+p$,	.......................,	$-(\nu_1+\nu+p)$,
$\nu_1+\nu+p+\sigma$,	$\nu_1+\nu+p+\sigma_1$,	$\sigma_1-\sigma$,
$\nu_1+\nu+p+\sigma$,	$\nu_1+\nu+p+\sigma+\sigma_1$,	σ_1,
$\nu_1+\nu+p+\sigma$,	$\nu_1+\nu+p+\sigma+\sigma_1$,	σ_1,
................,	.......................,	................,

and the degrees of the coefficients (z's) in (12) will be

$\nu_1+\nu+p$,	or $\nu_1+\nu+p$,
$\nu_1+\nu+p+\sigma_1$,	$\nu_1+\nu+p+\sigma$,
$\nu_1+\nu+p+\sigma+\sigma_1$,	$\nu_1+\nu+p+\sigma+\sigma_1$,
$\nu_1+\nu+p+\sigma+\sigma_1$,	$\nu_1+\nu+p+\sigma+\sigma_1$,
......................,	......................,

according as $\sigma_1 >$ or $< \sigma$.

The law has now been proved in the case of 1, 2, particular integrals; to prove it generally we must assume it for any given number (r), and show that it is true for $(r+1)$. Let p be the degree of X_n, and let the degrees of the succeeding coefficients rise by the quantities α, β, γ, ... where $\alpha > \beta > \gamma > \ldots.$ Then if μ, ν, σ be the degrees of M, N, $\phi(x)$ corresponding to the new integral to be introduced, we have

A,	B,	$B-A$,
$\nu+p$,	.......................,	$-(\nu+p)$,
$\nu+p+\alpha$,	$\nu+p+\sigma$,	$\sigma-\alpha$,
$\nu+p+\alpha+\beta$,	$\nu+p+\alpha+\sigma$,	$\sigma-\beta$,
$\nu+p+\alpha+\beta+\gamma$,	$\nu+p+\alpha+\beta+\sigma$,	$\sigma-\gamma$,
......................,	......................,	...........,

and if the value of σ lies between any two of the quantities α, β, γ, ..., *e.g.*, between β and γ, the degrees of the new coefficients will be

$$\begin{array}{l} \nu+p, \\ \nu+p+\alpha, \\ \nu+p+\alpha+\beta, \\ \nu+p+\alpha+\beta+\sigma, \\ \nu+p+\alpha+\beta+\sigma+\gamma, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

where $\alpha > \beta > \sigma > \gamma > \ldots.$

This law then holds good in every case of a differential equation formed with particular integrals as above described; so long as there is no accidental destruction of the highest powers of x in the formation of the Z's. The latter can, however, only take place when two or more of the quantities $\alpha, \beta, \gamma, \ldots \sigma$ are equal. Suppose that $h+1$ of these quantities become equal; then (there being no accidental destruction) there will be produced a series of $(h+1)$ coefficients, whose degrees rise by a constant difference. If there be an accidental destruction, it can take place only among the intermediate, not in the limiting, coefficients of this series. The result of all this is that, in such a case, a depression is to be considered as accidental, and may be replaced by a continuous ascent of as many steps as there are intervals between the limits of the depression.

From what has gone before we may represent geometrically the method to be employed: at equal distances on a given line draw a series of ordinates, the lengths of which are proportionate to the degrees of the coefficients of the given equation. Join the extremities, bridging over, however, any re-entering angles, and producing the ordinates so overstepped until they meet the bridge-line.

Suppose, for *example*, that the degrees of the coefficients are

$$1,\ 3,\ 4,\ 4,\ 4,\ 3,\ 2,\ 1,\ 0,$$

the equation being, of course, of the 8th degree. We then construct fig. 10.

The differences between the degrees of the coefficients are

$$2,\ 1,\ 0,\ 0,\ -1,\ -1,\ -1,\ -1.$$

And consequently the degrees of ψx, in the particular integrals $\varepsilon^{\psi(x)} Q$, will be

$$3,\ 2,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0.$$

Hence in the solution of the equation, we shall have

one integral of the form $\varepsilon^{\alpha x^2+\beta x^2+\gamma x} Q$,

.... $\varepsilon^{\alpha x^2+\beta x} Q$,

two .... $\varepsilon^{\alpha x} Q$,

four .... of a purely algebraic form.

As a *second example*, let the degrees of the coefficients be

$$1,\ 3,\ 3,\ 4,\ 4,\ 0,\ 2,\ 3,\ 2,\ 0.$$

Then we form the polygon, fig. 11. And after bridging over the re-entering angles, the differences of the degrees of the coefficients are

$$2,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ -\tfrac{1}{3},\ -\tfrac{1}{3},\ -\tfrac{1}{3},\ -1,\ -2,$$

and consequently the degrees of the exponentials $\psi(x)$ will be

$$3,\ \tfrac{3}{2},\ \tfrac{3}{2},\ 1,\ \tfrac{2}{3},\ \tfrac{2}{3},\ \tfrac{2}{3},\ 0,\ -1.$$

About the degree -1 there is a little difficulty; but the author suggests that the negative index arises from an accidental cancelling of the highest powers of x in z_0, and that it may probably be replaced by zero.

We now proceed to consider the indications of fractions in the particular integrals.

Let $$y_1 = \varepsilon^{\psi(x)} \frac{Q}{(x-a)^k},$$

where $\psi(x)$ is integral, and Q may be itself fractional. Then P_1 is of the form

$$P_1 = (x-a)^{-(k+n)} \{X_n Q_n + X_{n-1} Q_{n-1} (x-a) \\ + \ldots X_1 Q_1 (x-a)^{n-1} + X_0 Q_0 (x-a)^n\}.$$

If X_n, X_{n-1}, ... contain the factor $(x-a)$, then $(x-a)^p$ can be divided out of the numerator and denominator, and

$$P_1 = \frac{R}{(x-a)^q},$$

where $q < k+n$, and R has no factor $(x-a)$. Then

$$\frac{M}{N} = \frac{R'(x-a) - qR}{R(x-a)},$$

and N will contain $(x-a)$ once, and M will not.

It follows from a comparison with (12), that if X_n has no factor $(x-a)$, this factor will appear in the coefficient of the first term of the transformed equation. If X_n has such a factor, it will appear in the first and second coefficients and no others. Hence the presence of r integrals having $(x-a)$ to any powers in their denominators, will give rise to factors

$$(x-a)^r,\ (x-a)^{r-1},\ \ldots (x-a)$$

in the r first coefficients, and no more.

But as N contains all the factors of R we cannot conclude conversely from every factor $(x-a)$ in the first coefficient that there is a factor $(x-a)^k$ in the denominator of ϕ. But k can be so determined as to remove the uncertainty. Thus:

Let $$y=\frac{Q}{(x-a)^k},$$

where Q may contain an exponential. Then

$$y'=\frac{Q'(x-a)-kQ}{(x-a)^{k+1}}=\frac{Q_1}{(x-a)^{k+1}},$$

$$y''=\frac{Q_1'(x-a)-(k+1)Q_1}{(x-a)^{k+2}}=\frac{Q_2}{(x-a)^{k+2}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$y^{(n)}=\frac{Q'_{n-1}(x-a)-(k+n-1)Q_{n-1}}{(x-a)^{k+n}}=\frac{Q_n}{(x-a)^{k+n}},$$

whence identically,

$$X_n\frac{Qn}{x-a}+X_{n-1}Q_{n-1}+X_{n-2}Q_{n-2}(x-a)+\ldots X_0Q(x-a)^{n-1}=0,$$

and as this holds good for all values of x, let $x=a_1$; then

$$\left\{X_n\frac{Q_n}{x-a}+X_{n-1}Q_{n-1}\right\}_{x=a}=0,$$

But $$Q_n=Q'_{n-1}(x-a)-(k+n-1)Q_{n-1};$$

therefore $$Q_n\{_a=-(k+n-1)\,Q_{n-1}\}_{x=a};$$

therefore $$\left\{-\frac{X_n}{x-a}(k+n-1)+X_{n-1}\right\}_{x=a}=0;$$

therefore $$\left\{\frac{X_{n-1}}{X_n}(x-a)\right\}_{x=a}=\frac{X_{n-1}}{X_n'}=k+(n-1),\quad k=\frac{X_{n-1}}{X_n'}-(n-1),$$

which gives the value of the exponent of $(x-a)$ in the denominator of the value of y, when $x-a$ appears as factor of X_n.

The same process followed for 2, 3, ... particular integrals containing $(x-a)$ to any powers in their denominators, leads to a quadratic, cubic, ... for determining k.

Secondly, suppose

$$y_1=\varepsilon^{\int\frac{\phi(x)}{(x-a)^m}dx}=\varepsilon^{\Phi},$$

then

$$y_1 = \varepsilon^{\Phi},\ y_1' = \varepsilon^{\Phi} \frac{Q_1}{(x-a)^m},\ y'' = \varepsilon^{\Phi} \frac{Q_2}{(x-a)^{2m}},\ \ldots y^{(n)} = \varepsilon^{\Phi} \frac{Q_n}{(x-a)^{nm}},$$

$$P_1 = \varepsilon^{\Phi} (x-a)^{nm} \{X_n Q_n + X_{n-1} Q_{n-1}(x-a) + \ldots X_0 (x-a)^{nm}\}.$$

If any of X_n, X_{n-1}, ... contain the factor $(x-a)$, we can divide it out and have

$$P_1 = \varepsilon^{\Phi} \frac{R}{(x-a)^{\mu}},$$

$$\frac{P_1'}{P_1} = \frac{\phi(x)}{(x-a)^m} + \frac{R'}{R} - \frac{\mu}{x-a},\quad \frac{M}{N} = \frac{T}{(x-a)^m},\quad \text{or } \frac{T}{x-a},$$

according as m, always supposed positive, is not less than 1, or is less than 1.

From this the author concludes that:

(1) The introduction of a particular integral of the above form into an equation not having the factor $(x-a)$ in its coefficients, produces the factor $(x-a)$ in the first coefficient, and generally in no others.

(2) The introduction of a second, with $(x-a)^p$ in the denominator, gives rise to the factor $(x-a)$ in the second coefficient also. If the first coefficient of (1) contains the factor $(x-a)^m$, the first of (12) will contain $(x-a)^{m+p}$, and the second either $(x-a)^m$ or $(x-a)^p$ according as $m<p$, or $p<m$.

And as an example, let the coefficient

$$X_n,\ X_{n-1},\ X_{n-2},\ X_{n-3},\ X_{n-4},\ X_{n-5},\ X_{n-6},\ X_{n-7},\ X_{n-8},$$

contain $(x-a)$ to the powers

$$8,\ 6,\ 5,\ 7,\ 2,\ 2,\ 1,\ 3,\ 0,$$

then drawing the polygon (fig. 12) and cutting off the *projecting* angles, we have, for the series of differences of degrees,

$$2,\ \tfrac{4}{3},\ \tfrac{4}{3},\ \tfrac{4}{3},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2};$$

hence we conclude one integral of the form

$$\varepsilon^{\int \frac{\phi(x)}{(x-a)^2} dx},$$

and three of the form

$$\varepsilon^{\int \frac{\phi(x)}{(x-a)^{\frac{4}{3}}} dx},$$

but, $\frac{1}{2}$ being less than unity, we cannot conclude about the rest.

GEOMETRICAL INVESTIGATION OF CERTAIN TRIGONOMETRICAL FORMULÆ.

By J. Bond, B.A., Fellow of St. Mary Magdalene College, Cambridge.

LET O (fig. 13) be the centre of the circumscribed circle of the triangle ABC. r, R the radii of the inscribed and circumscribed circles.

By Euclid VI. D.

$$OD.EC + OE.CD = DE.OC \,\ldots\ldots\ldots\ldots(1);$$

therefore $$\frac{CD}{OD} + \frac{CE}{OE} = \frac{R}{2}\cdot\frac{AB}{OD.OE}.$$

Similarly $$\frac{CE}{OE} + \frac{CF}{OF} = \frac{R}{2}\cdot\frac{BC}{OE.OF},$$

$$\frac{CF}{OF} + \frac{CD}{OD} = \frac{R}{2}\cdot\frac{CA}{OF.OD};$$

therefore $$2\left\{\frac{CD}{OD} + \frac{CE}{OE} + \frac{CD}{OD}\right\}$$

$$= \frac{R}{2\,OD.OE.OF}\{AB.OF + BC.OD + CA.OE\}$$

$$= \frac{EG.AH.BC}{4\,OD.OE.OF}$$

$$= \frac{FE.CA.AB}{4\,OD.OE.OF} = 2\,\frac{CD}{OD}\cdot\frac{CE}{OE}\cdot\frac{AF}{OF};$$

therefore $\tan A + \tan B + \tan C = \tan A . \tan B . \tan C.$

Again, equation (1) may be written

$$OD.AC + OE.BC = AB.R.$$

Similarly $$OE.AB + OF.AC = BC.R,$$

$$OF.BC + OD.AB = CA.R,$$

and $OD.BC + OE.CA + OF.AB = r\{AB + BC + CA\}$;

therefore adding

$$(OD + OE + OF)(AB + BC + CA) = (R + r)(AB + BC + CA),$$

or $$OD + OE + OF = R + r.$$

ON CERTAIN PROPERTIES OF THE TETRAHEDRON.

By N. M. Ferrers.

Let abc be any triangle, then the determinant

$$\begin{vmatrix} 0, & 1, & 1 & 1 \\ 1, & 0^2, & ab^2, & ac^2 \\ 1, & ba^2, & 0, & bc^2 \\ 1, & ca^2, & cb^2, & 0 \end{vmatrix}$$

together with its several minors, possess important properties with respect to the triangle. For if A be its area, ρ the radius of the circumscribed circle, we know that

$$(2A)^2 = \tfrac{1}{4}\left(2ab^2.ac^2 + 2bc^2.ba^2 + 2ca^2.cb^2 - bc^4 - ca^4 - ab^4\right)$$

$$= -\tfrac{1}{4}\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & ab^2, & ac^2 \\ 1, & ba^2, & 0, & bc^2 \\ 1, & ca^2, & cb^2, & 0 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, $(2\rho A)^2 = \tfrac{1}{4}bc^2.ca^2.ab^2 = \tfrac{1}{8}\begin{vmatrix} 0, & ab^2, & ac^2 \\ ba^2, & 0, & bc^2 \\ ca^2, & cb^2, & 0 \end{vmatrix} \ldots\ldots\ldots(2),$

which interprets the minor formed by omitting the first line and column. If we omit any two other corresponding lines and columns, (say the second) we get

$$\begin{vmatrix} 0, & 1, & 1 \\ 1, & 0, & bc^2 \\ 1, & cb^2, & 0 \end{vmatrix} = 2.bc^2 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

or twice the square on a side.

Omitting the second line and third column, we get

$$\begin{vmatrix} 0, & 1, & 1 \\ 1, & ba^2, & bc^2 \\ 1, & ca^2, & 0 \end{vmatrix} = ca^2 + bc^2 - ab^2 = 2bc.ca \cos c \ldots..(4),$$

or, twice the product of two sides into the cosine of the angle between them.

Lastly, omitting the first line and second column, we have

$$\begin{vmatrix} 1, & ab^2, & ac^2 \\ 1, & 0, & bc^2 \\ 1, & cb^2, & 0 \end{vmatrix} = bc^2\,(ca^2 + ab^2 - bc^2) = 2bc.ca.ab.\frac{ca^2 + ab^2 - bc^2}{2ca.ab}.bc$$
$$= \tfrac{1}{2}\rho A.\cos a.bc$$
$$= \tfrac{1}{2}A.bc.\rho\,\cos a \ldots\ldots\ldots(5),$$

or half the area of the triangle, into the product of a side and the distance of the circumscribed circle from that side. This will be $(2A)^2.\frac{\bar{\alpha}}{2}$, if $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be the triangular coordinates of the centre.

Our object is to demonstrate properties analogous to the foregoing, for the determinant formed in a similar manner from the tetrahedron *abcd*.

We shall denote the areas of the faces respectively opposite to the angular points a, b, c, d by A, B, C, D; the mutual inclination of the faces A, B by $\widehat{AB}$, the volume of the tetrahedron by V, and the radius of the circumscribed sphere by R.

Many of the following results are already known, but they are inserted here for the sake of completeness.

The tetrahedral equation of the circumscribed sphere is

$$\Sigma\,(ab^2.\alpha\beta) = 0.$$

Hence, if $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ be the coordinates of the centre, we have

$$\left.\begin{array}{l} ab^2.\bar{\beta} + ac^2.\bar{\gamma} + ad^2.\bar{\delta} = S, \text{ say} \\ ba^2.\bar{\alpha} + bc^2.\bar{\gamma} + bd^2.\bar{\delta} = S \\ ca^2.\bar{\alpha} + cb^2.\bar{\beta} + cd^2.\bar{\delta} = S \\ da^2.\bar{\alpha} + db^2.\bar{\beta} + dc^2.\bar{\gamma} = S \\ \bar{\alpha} + \bar{\beta} + \bar{\gamma} + \bar{\delta} = 1 \end{array}\right\} \ldots\ldots\ldots(\text{i}).$$

Multiplying the first four of these equations in order by $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$, and adding, we have

$$2\Sigma\,(ab^2.\bar{\alpha}\,\bar{\beta}) = S.$$

Again, $$R^2 + \Sigma\,\{ab^2\,(\alpha - \bar{\alpha})\,(\beta - \bar{\beta})\} = 0,$$

or $$R^2 + \Sigma\,(ab^2\alpha\beta + ab^2\bar{\alpha}\,\bar{\beta}) - (ab^2\bar{\beta} + ac^2\bar{\gamma} + ad^2\bar{\delta})\,\alpha$$
$$- (bc^2\bar{\gamma} + bd^2\bar{\delta} + ba^2\bar{\alpha})\,\beta - (cd^2\bar{\delta} + ca^2\bar{\alpha} + cb^2\bar{\beta})\,\gamma$$
$$- (da^2\bar{\alpha} + db^2\bar{\beta} + dc^2\bar{\gamma})\,\delta = 0,$$

whence, since $\Sigma\,(ab^2\alpha\beta) = 0$, $\Sigma\,(ab^2\bar{\alpha}\bar{\beta}) = \frac{1}{2}S$,

and $$\alpha+\beta+\gamma+\delta=1,$$

$$R^2 - \tfrac{1}{2}S = 0;$$

therefore $$R^2 = \Sigma\,(ab^2.\bar{\alpha}\bar{\beta}),$$

and, from equations (1),

$$R^2 = \tfrac{1}{2}S = \frac{\tfrac{1}{2}\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2 \\ ba^2, & 0, & bc^2, & bd^2 \\ ca^2, & cb^2, & 0, & cd^2 \\ da^2, & db^2, & dc^2, & 0 \end{vmatrix}}{\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & ab^2, & ac^2, & ad^2 \\ 1, & ba^2, & 0, & bc^2, & bd^2 \\ 1, & ca^2, & cb^2, & 0, & cd^2 \\ 1, & da^2, & db^2, & dc^2, & 0 \end{vmatrix}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Again, from equations (1), it appears that

$$\frac{\bar{\alpha}}{\begin{vmatrix} 1, & ab^2, & ac^2, & ad^2 \\ 1, & 0, & bc^2, & bd^2 \\ 1, & cb^2, & 0, & cd^2 \\ 1, & db^2, & dc^2, & 0 \end{vmatrix}} = \frac{\bar{\beta}}{\begin{vmatrix} 1, & bc^2, & bd^2, & ba^2 \\ 1, & 0, & cd^2, & ca^2 \\ 1, & dc^2, & 0, & da^2 \\ 1, & ac^2, & ad^2, & 0 \end{vmatrix}} = \frac{\bar{\gamma}}{\begin{vmatrix} 1, & cd^2, & ca^2, & cb^2 \\ 1, & 0, & da^2, & db^2 \\ 1, & ad^2, & 0, & ab^2 \\ 1, & bd^2, & ba^2, & 0 \end{vmatrix}}$$

$$= \frac{\bar{\delta}}{\begin{vmatrix} 1, & da^2, & db^2, & dc^2 \\ 1, & 0 & ab^2, & ac^2 \\ 1, & ba^2, & 0, & bc^2 \\ 1, & ca^2, & cb^2, & 0 \end{vmatrix}} = \frac{S}{\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2 \\ ba^2, & 0, & bc^2, & bd^2 \\ ca^2, & cb^2, & 0, & cd^2 \\ da^2, & db^2, & dc^2, & 0 \end{vmatrix}} \quad \ldots\ldots (7).$$

Now $$(6V)^2 = ab^2.ac^2.ad^2 \begin{vmatrix} 1, & \cos bac, & \cos bad \\ \cos cab, & 1, & \cos cad \\ \cos dab, & \cos dac, & 1 \end{vmatrix}.$$

$$= \tfrac{1}{8}\begin{vmatrix} 2ab^2, & ab^2+ac^2-bc^2, & ab^2+ad^2-bd^2 \\ ac^2+ab^2-bc^2, & 2ac^2, & ac^2+ad^2-cd^2 \\ ad^2+ab^2-bd^2, & ad^2+ac^2-cd^2, & 2ad^2 \end{vmatrix}$$

and it has been proved by Mr. Salmon (*Journal*, t. III., p. 282) that when this determinant vanishes, the denominator of R^2 (in equation (6)) does so likewise.

These two determinants, since they are of the same dimensions, can therefore only differ by a numerical factor, and they may be ascertained, by comparing the coefficients of any term, (ab^2cd^4 for instance) to be identical. Hence

$$(6V)^2 = \tfrac{1}{8} \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & ab^2, & ac^2, & ad^2 \\ 1, & ba^2, & 0, & bc^2, & bd^2 \\ 1, & ca^2, & cb^2, & 0, & cd^2 \\ 1, & da^2, & db^2, & dc^2, & 0 \end{vmatrix} \quad \ldots\ldots\ldots (8).$$

This is the extension of equation (1).

By equation (6) it appears that

$$(6RV)^2 = \tfrac{1}{16} \begin{vmatrix} 0, & ab^2, & ac^2, & ad^2 \\ ba^2, & 0, & bc^2, & bd^2 \\ ca^2, & cb^2, & 0, & cd^2 \\ da^2, & db^2, & dc^2, & 0 \end{vmatrix} \quad \ldots\ldots\ldots (9),$$

which is the extension of (2).

We have at once

$$(2A)^2 = -\tfrac{1}{4} \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & bc^2, & bd^2 \\ 1, & cb^2, & 0, & cd^2 \\ 1, & db^2, & dc^2, & 0 \end{vmatrix} \quad \ldots\ldots\ldots\ldots (10),$$

the extension of (3).

And by equations (7),

$$(6V)^2 \frac{\bar{\alpha}}{2} = \begin{vmatrix} 1, & ab^2, & ac^2, & ad^2 \\ 1, & 0, & bc^2, & bd^2 \\ 1, & cb^2, & 0, & cd^2 \\ 1, & db^2, & dc^2, & 0 \end{vmatrix} \quad \ldots\ldots\ldots\ldots (11),$$

the extension of (5).

Lastly, suppose a sphere to be described about a as centre, cutting ab, ac, ad in p, q, r, respectively. Then $\widehat{CD} = qpr$, and

$$\cos qpr = \frac{\cos qr - \cos pq \,.\, \cos pr}{\sin pq \,.\, \sin pr}$$

$$= \frac{ab^2 . ac . ad \cos qr - ab . ac \cos pq . ab . ad \cos pr}{ab . ac \sin pq . ab . ad \sin pr}$$

$$= \frac{2ab^2(ac^2 + ad^2 - cd^2) - (ab^2 + ac^2 - bc^2)(ab^2 + ad^2 - bd^2)}{16\,CD}.$$

Hence $16CD\cos\widehat{CD} = \begin{vmatrix} 2ab^2, & ab^2+ac^2-bc^2 \\ ab^2+ad^2-bd^2, & ac^2+ad^2-cd^2 \end{vmatrix}.$

Now, suppose x, y, z to be three quantities which satisfy the four equations,

$$\begin{aligned} x + y + z &= 0, \\ 1 + ab^2.y + ac^2.z &= 0, \\ 1 + ba^2.x + bc^2.z &= 0, \\ 1 + da^2.x + db^2.y + dc^2.z &= 0. \end{aligned}$$

The condition for this, is

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & ab^2, & ac^2 \\ 1, & ba^2, & 0, & bc^2 \\ 1, & da^2, & db^2, & dc^2 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (12).$$

But multiplying the first of the above equations by ab^2, the second by 1, the third by -1, and adding, we get

$$2ab^2.y + (ab^2 + ac^2 - bc^2)\,z = 0,$$

and multiplying the first by ad^2, the second by 1, and the fourth by -1, and adding,

$$(ad^2 + ab^2 - bd^2)\,y + (ad^2 + ac^2 - cd^2)\,z = 0.$$

Eliminating y and z between these two equations, we get

$$\begin{vmatrix} 2ab^2, & ab^2 + ac^2 - bc^2 \\ ad^2 + ab^2 - bd^2, & ad^2 + ac^2 - cd^2 \end{vmatrix} = 0 \ldots\ldots (13).$$

Hence, the left-hand members of (12) and (13) vanish together, and they may be ascertained, by comparing the coefficients of any term, to be identical. We thus see that

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & ab^2, & ac^2 \\ 1, & ba^2, & 0, & bc^2 \\ 1, & da^2, & db^2, & dc^2 \end{vmatrix} = 16\,CD\cos\widehat{CD} \ldots\ldots\ldots (14),$$

or, the determinant formed by omitting any line and column, (not the first) is equal to sixteen times the product of the two opposite faces into the cosine of the angle between them. This (which in fact includes (12)) is the extension of (4).

We have thus completed the discussion of the first minors of the determinant. The properties may no doubt be more elegantly investigated.

We may complete this investigation by briefly considering the second minors.

It is, however, needless to consider any case in which a corresponding line and column are omitted, except the first, as these, which expunge one letter altogether, will have been already considered under the first minors of the triangle abc.

Omitting the first and second line, and first and third column, we get

$$\begin{vmatrix} ba^2, & bc^2, & bd^2 \\ ca^2, & 0, & cd^2 \\ da^2, & dc^2, & 0 \end{vmatrix} = cd^2(-ba^2.cd^2 + ca^2.bd^2 + da^2.bc^2),$$

a result somewhat analogous to the expression for the cosine of the angle of a triangle.

Omitting the first and second line, and third and fourth column,

$$\begin{vmatrix} 1, & ba^2, & bd^2 \\ 1, & ca^2, & cd^2 \\ 1, & da^2, & 0 \end{vmatrix} \begin{aligned} &= -cd^2.da^2 + da^2.bd^2 + ba^2.bd^2 - ca^2.bd^2, \\ &= bd^2(ad^2 + cd^2 - ca^2) - cd^2(da^2 + bd^2 - ba^2), \\ &= 2ad.bd.cd(bd \cos abd - cd \cos adb). \end{aligned}$$

Omitting the second and third line, and fourth and fifth column,

$$\begin{vmatrix} 0, & 1, & 1 \\ 1, & ca^2, & cb^2 \\ 1, & da^2, & db^2 \end{vmatrix} = ad^2 + bc^2 - ac^2 - bd^2,$$

which has been shewn (*Journal*, t. III., p. 145, or Frost and Wolstenholme's *Solid Geometry*, p. 25, Ex. 16) to be equal to $2ab.cd.\cos(ab, cd)$, or twice the product of two opposite edges into the cosine of their mutual inclination.

August 14, 1861.

NOTES ON TETRAHEDRAL AND QUADRIPLANAR COORDINATES.

By N. M. Ferrers.

Suppose (e, f, g, h) to be the tetrahedral coordinates of a point, and

$$k\alpha + l\beta + m\gamma + n\delta = 0 \text{..................}(1),$$

the tetrahedral equation of a plane, and that it is required to find the distance between them. If $e+\xi$, $f+\eta$, $g+\zeta$, $h+\theta$, be the coordinates of any point in (1), we shall have

$$\xi + \eta + \zeta + \theta = 0 \text{..............................}(2),$$

$$k\xi + l\eta + m\zeta + n\theta = -(ke + lf + mg + nh) \text{......}(3),$$

and if ν be the distance between these points,

$$\Sigma\,(ab^2\xi\eta) = -\,r^2 \quad\text{......................}(4).$$

To find the distance from the point to the plane, we must make ν a minimum, subject to the condition (2) and (3).

This will give, multiplying (2) by the indeterminate multiplier λ, and (3) by μ,

$$\begin{aligned}
ab^2.\eta + ac^2.\zeta + ad^2.\theta + \lambda + \mu k &= 0,\\
ba^2.\xi + bc^2.\zeta + bd^2.\theta + \lambda + \mu l &= 0,\\
ca^2.\xi + cb^2.\eta + cd^2.\theta + \lambda + \mu m &= 0,\\
da^2.\xi + db^2.\eta + dc^2.\zeta + \lambda + \mu n &= 0.
\end{aligned}$$

Multiplying these in order by ξ, η, ζ, θ, and adding, we get

$$2r^2 + \mu\,(ke + lf + mg + nh) = 0.$$

Substituting this value for μ, and eliminating ξ, η, ζ, θ, λ between (2) and (3) and the above four equations, we get

$$r^2 = \frac{\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1\\ ba^2, & 0, & bc^2, & bd^2, & 1\\ ca^2, & cb^2, & 0, & cd^2, & 1\\ da^2, & db^2, & dc^2, & 0, & 1\\ 1, & 1, & 1, & 1, & 0\end{vmatrix}}{\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1, & k\\ ba^2, & 0, & bc^2, & bd^2, & 1, & l\\ ca^2, & cb^2, & 0, & cd^2, & 1, & m\\ da^2, & db^2, & dc^2, & 0, & 1, & n\\ 1, & 1, & 1, & 1, & 0, & 0\\ k, & l, & m, & n, & 0, & 0\end{vmatrix}}\,(ke + lf + mg + nh)^2,$$

or, as it may also be written,

$$r^2 = \frac{18\,(ke + lf + mg + nh)^2.\,V^2}{(kA)^2 + (lB)^2 + (mC)^2 + (nD)^2 - 2\Sigma\,(klAB\cos\widehat{AB})},$$

A, B, C, D being the areas of the faces of the tetrahedron, and V its volume.

The expression in quadriplanar coordinates, where e, f, g, h represent the distances from the respective faces, will be

$$r^2 = \frac{(ke + lf + mg + nh)^2}{k^2 + l^2 + m^2 + n^2 - 2\Sigma\,(kl\cos\widehat{AB})}.$$

We may hence deduce the relation among the distances of a plane from four given points. Suppose t, u, v, w to be these distances, and take the points as angular points of the tetrahedron of reference. We shall then have successively,

$$e=1,\ f=0,\ g=0,\ h=0,\ r=t,$$

$$e=0,\ f=1,\ g=0,\ h=0,\ r=u,$$

and so on. Hence

$$\frac{t^2}{k^2}=\frac{u^2}{l^2}=\frac{v^2}{m^2}=\frac{w^2}{n^2}.$$

whence, substituting in the general value of r^2,

$$\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1, & t \\ ba^2, & 0, & bc^2, & bd^2, & 1, & u \\ ca^2, & cb^2, & 0, & cd^2, & 1, & v \\ da^2, & db^2, & dc^2, & 0, & 1, & w \\ 1, & 1, & 1, & 1, & 0, & 0 \\ t, & u, & v, & w, & 0, & 0 \end{vmatrix} = \begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1 \\ ba^2, & 0, & bc^2, & bd^2, & 1 \\ ca^2, & cb^2, & 0, & cd^2, & 1 \\ da^2, & db^2, & dc^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

or

$$\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1, & t \\ ba^2, & 0, & bc^2, & bd^2, & 1, & u \\ ca^2, & cb^2, & 0, & cd^2, & 1, & v \\ da^2, & db^2, & dc^2, & 0, & 1, & w \\ 1, & 1, & 1, & 1, & 0, & 0 \\ t, & u, & v, & w, & 0, & -1 \end{vmatrix} = 0,$$

the required relation, analogous to that given in Salmon's *Higher Plane Curves*, between the distances of a straight line from three given points, which may be expressed in a similar form.

We may also obtain the equations of the eight spheres touching the four faces of the tetrahedron of reference. Thus, to find the inscribed sphere, we consider it as the envelope of a plane, whose distance from a given point is constant.

Using quadriplanar coordinates, the coordinates of this point are all equal to one another, and to its distance from the plane, we have therefore to find the envelope of the plane

$$k\alpha + l\beta + m\gamma + n\delta = 0 \text{ ....(i)},$$

subject to the condition

$$(k+l+m+n)^2 = k^2 + l^2 + m^2 + n^2 - 2\Sigma\,(kl\cos \widehat{AB}),$$

or $$\Sigma\left(kl\cos^2\frac{\hat{AB}}{2}\right)=0 \text{ ....(ii)}.$$

Multiplying (i) by an indeterminate multiplier λ, adding it to (2), differentiating, and equating to zero the coefficients of dk, dl, dm, dn, we get

$$\begin{aligned}
& l\cos^2\frac{\hat{AB}}{2}+m\cos^2\frac{\hat{AC}}{2}+n\cos^2\frac{\hat{AD}}{2}+\lambda\alpha=0,\\
k\cos^2\frac{\hat{BA}}{2} & \qquad\qquad +m\cos^2\frac{\hat{BC}}{2}+n\cos^2\frac{\hat{BD}}{2}+\lambda\beta=0,\\
k\cos^2\frac{\hat{CA}}{2}+l\cos^2\frac{\hat{CB}}{2} & \qquad\qquad +n\cos^2\frac{\hat{CD}}{2}+\lambda\gamma=0,\\
k\cos^2\frac{\hat{DA}}{2}+l\cos^2\frac{\hat{DB}}{2}+m\cos^2\frac{\hat{DC}}{2} & \qquad\qquad +\lambda\delta=0.
\end{aligned}$$

Eliminating k, l, m, n, λ between these equations and (1), we get, as the equation of the inscribed sphere,

$$\begin{vmatrix}
0, & \alpha, & \beta, & \gamma, & \delta\\
\alpha, & 0, & \cos^2\frac{\hat{AB}}{2}, & \cos^2\frac{\hat{AC}}{2}, & \cos^2\frac{\hat{AD}}{2}\\
\beta, & \cos^2\frac{\hat{BA}}{2}, & 0, & \cos^2\frac{\hat{BC}}{2}, & \cos^2\frac{\hat{BD}}{2}\\
\gamma, & \cos^2\frac{\hat{CA}}{2}, & \cos^2\frac{\hat{CB}}{2}, & 0, & \cos^2\frac{\hat{CD}}{2}\\
\delta, & \cos^2\frac{\hat{DA}}{2}, & \cos^2\frac{\hat{DB}}{2}, & \cos^2\frac{\hat{DC}}{2}, & 0
\end{vmatrix}=0.$$

which agrees with the form given by Mr. Salmon in this *Journal*, (t. IV., p. 271).

For the sphere touching A externally and the other three faces produced, we have to find the envelop of (i) subject to the condition

$$(-k+l+m+n)^2=k^2+l^2+m^2+n^2-2\Sigma(kl\cos\hat{AB}),$$

or $$-k\left(l\sin^2\frac{\hat{AB}}{2}+m\sin^2\frac{\hat{AC}}{2}+n\sin^2\frac{\hat{AD}}{2}\right)$$
$$+mn\cos^2\frac{\hat{CD}}{2}+nl\cos^2\frac{\hat{DB}}{2}+lm\cos^2\frac{\hat{BC}}{2}=0.$$

The equation of this sphere will therefore be

$$\begin{vmatrix} 0 & -\alpha, & \beta, & \gamma, & \delta \\ -\alpha, & 0, & \sin^2\frac{\widehat{AB}}{2}, & \sin^2\frac{\widehat{AC}}{2}, & \sin^2\frac{\widehat{AD}}{2} \\ \beta, & \sin^2\frac{\widehat{BA}}{2}, & 0, & \cos^2\frac{\widehat{BC}}{2}, & \cos^2\frac{\widehat{BD}}{2} \\ \gamma, & \sin^2\frac{\widehat{CA}}{2}, & \cos^2\frac{\widehat{CB}}{2}, & 0, & \cos^2\frac{\widehat{CD}}{2} \\ \delta, & \sin^2\frac{\widehat{DA}}{2}, & \cos^2\frac{\widehat{DB}}{2}, & \cos^2\frac{\widehat{DC}}{2}, & 0 \end{vmatrix} = 0.$$

For the sphere touching A and B on the positive, and C and D on the negative side, (or *vice versâ* as the case may be) we require the envelope of (i) subject to the condition

$$(k+l-m-n)^2 = k^2+l^2+m^2+n^2-2\Sigma\,(kl\cos\widehat{AB}),$$

$$\text{or } kl\cos^2\frac{\widehat{AB}}{2}+mn\cos^2\frac{\widehat{CD}}{2}-km\sin^2\frac{\widehat{AC}}{2}$$

$$-ln\sin^2\frac{\widehat{BD}}{2}-kn\sin^2\frac{\widehat{AD}}{2}-lm\sin^2\frac{\widehat{BC}}{2}=0,$$

which will give, for the equation of this sphere,

$$\begin{vmatrix} 0, & \alpha, & \beta, & \gamma, & \delta \\ \alpha, & 0, & \cos^2\frac{\widehat{AB}}{2}, & -\sin^2\frac{\widehat{AC}}{2}, & -\sin^2\frac{\widehat{AD}}{2} \\ \beta, & \cos^2\frac{\widehat{BA}}{2}, & 0, & -\sin^2\frac{\widehat{BC}}{2}, & -\sin^2\frac{\widehat{BD}}{2} \\ \gamma, & -\sin^2\frac{\widehat{CA}}{2}, & -\sin^2\frac{\widehat{CB}}{2}, & 0, & \cos^2\frac{\widehat{CD}}{2} \\ \delta, & -\sin^2\frac{\widehat{DA}}{2}, & -\sin^2\frac{\widehat{DB}}{2}, & \cos^2\frac{\widehat{DC}}{2}, & 0 \end{vmatrix} = 0.$$

August 14, 1861.

DETERMINATION OF THE FOCI OF THE CONIC SECTION EXPRESSED BY TRILINEAR COORDINATES.

By P. J. Hensley, B.A., Fellow of Christ's College, Cambridge.

Suppose that α, β, γ are trilinear coordinates.

Equations determining the coordinates of the foci of the curve represented by the equation

$$l\alpha^2 + m\beta^2 + n\gamma^2 + 2l'\beta\gamma + 2m'\gamma\alpha + 2n'\alpha\beta = 0 \ \ldots\ldots(1),$$

may be found by means of the following property: If from a focus of a conic section perpendiculars be drawn to a pair of parallel tangents, the rectangle contained by them is constant.

We may by the application of this property to three pairs of tangents obtain a sufficient number of equations to determine the foci; it will be convenient for this purpose to consider the three pairs of tangents parallel to the sides of the fundamental triangle.

Suppose then that the equation

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

represents a tangent to (1). Then if γ be eliminated between (1) and (2), the resulting equation must give equal values for the ratio $\alpha : \beta$.

The condition arising from this will be found to be

$$(mn - l'^2)\lambda^2 + (nl - m'^2)\mu^2 + (lm - n'^2)\nu^2$$
$$+ 2(m'n' - ll')\mu\nu + 2(n'l' - mm')\nu\lambda + 2(l'm' - nn')\lambda\mu = 0 \ldots(3).$$

Or

$$\begin{vmatrix} l, & n', & m', & \lambda \\ n', & m, & l', & \mu \\ m', & l', & n, & \nu \\ \lambda, & \mu, & \nu, & 0 \end{vmatrix} = 0.$$

Now the equation to a line parallel to the straight line $(\alpha = 0)$ is

$$\lambda\alpha + b\beta + c\gamma = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

if a, b, c are the sides of the fundamental triangle.

Or, if the area of the triangle be denoted by $\frac{1}{2}H$,

$$(\lambda - a)\,\alpha + H = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

This by (3) will be a tangent to the conic if λ satisfies the equation

$$(mn - l'^2)\,\lambda^2 + 2\,\{(l'm' - nn')\,b + (n'l' - mm')\,c\}\,\lambda$$
$$+ (nl - m'^2)\,b^2 + (lm - n'^2)\,c^2 + 2\,(m'n' - ll')\,bc = 0 \ldots (6).$$

And the distances of the two parallel tangents from the side a will be given by the equation obtained by substituting in (6) the value of λ given by (5).

This equation is

$$\begin{vmatrix} l, & n', & m', & a \\ n', & m, & l', & b \\ m', & l', & n, & c \\ a, & b, & c, & 0 \end{vmatrix} \alpha^2 - 2 \begin{vmatrix} n', & m', & a \\ m, & l', & b \\ l', & n, & c \end{vmatrix} H\alpha + \begin{vmatrix} m, & l' \\ l', & n \end{vmatrix} H^2 = 0.$$

If the first determinant in this equation be denoted by P, it may be written in the form

$$P\alpha^2 - \frac{dP}{da}\,H\alpha + \frac{1}{2}\,\frac{d^2P}{da^2}\,H^2 = 0 \;\ldots\ldots\ldots\ldots\; (7).$$

Now if α_1 and α_2 be the roots of this equation, and α the perpendicular from a focus on the line $\alpha = 0$; the perpendiculars on the two tangents from the focus are respectively $\alpha - \alpha_1$ and $\alpha - \alpha_2$; and the rectangle contained by them is $\alpha^2 - (\alpha_1 + \alpha_2)\,\alpha + \alpha_1\alpha_2$.

And similar quantities for the other sides being denoted by similar symbols, we have the relations

$$\alpha^2 - (\alpha_1 + \alpha_2)\,\alpha + \alpha_1\alpha_2 = \beta^2 - (\beta_1 + \beta_2)\,\beta + \beta_1\beta_2 = \gamma^2 - (\gamma_1 + \gamma_2)\,\gamma + \gamma_1\gamma_2.$$

And by means of (7) and the corresponding equations,

$$P\alpha^2 - \frac{dP}{da}\,H\alpha + \frac{1}{2}\,\frac{d^2P}{da^2}\,H^2$$
$$= P\beta^2 - \frac{dP}{db}\,H\beta + \frac{1}{2}\,\frac{d^2P}{db^2}\,H^2 = P\gamma^2 - \frac{dP}{dc}\,H\gamma + \frac{1}{2}\,\frac{d^2P}{dc^2}\,H^2 \ldots (8).$$

These are the equations which give the foci of the conic represented by (1). From these may of course be deduced the equations for the special forms

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,\; l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,\; \sqrt{(l\alpha)} \pm \sqrt{(m\beta)} \pm \sqrt{(n\gamma)} = 0.$$

For the last of these, however, a direct investigation is more simple.

It may be shewn that if (2) is a tangent to this curve, then

$$\frac{l}{\lambda}+\frac{m}{\mu}+\frac{n}{\nu}=0.$$

So that for the tangent parallel to the line $(\alpha=0)$, which is itself a tangent,

$$\frac{l}{\lambda}+\frac{m}{b}+\frac{n}{c}=0,$$

and by (5) the distance between this pair of tangents is $\frac{aQ-l}{a^2\alpha}H$, where Q stands for the quantity $\frac{l}{a}+\frac{m}{b}+\frac{n}{c}$.

Consequently, if α, β, γ are coordinates of a focus,

$$\alpha\left\{\alpha-\frac{aQ-l}{a^2Q}H\right\}=\beta\left\{\beta-\frac{bQ-m}{b^2Q}H\right\}=\gamma\left\{\gamma-\frac{cQ-n}{c^2Q}H\right\}\dots(9).$$

Equations for the foci of the conic $k\gamma^2-\alpha\beta=0$ may be deduced from (8), or may be found as follows:

Since (figs. 14, 15) the sides CA, CB are tangents, if S is a focus, the angles ASC, BSC are either equal or supplemental. In either case, if CS meet AB in m,

$$Am : Bm :: AS : BS.$$

If α, β, γ are trilinear coordinates of S, then

$$Am : Bm :: b\beta : a\alpha,$$

and if the angle $CBS=\theta$,

$$\alpha=BS\sin\theta,\quad \gamma=BS\sin(B-\theta),$$

whence, eliminating θ,

$$BS^2\sin^2 B=\alpha^2+\gamma^2+2\alpha\gamma\cos B,$$

and similarly for AS and by the above proportion

$$(\alpha^2+\gamma^2+2\alpha\gamma\cos B)\,\beta^2=(\beta^2+\gamma^2+2\beta\gamma\cos A)\,\alpha^2.$$

Or, rejecting the factor γ,

$$(\alpha^2-\beta^2)\,\gamma+2\alpha\beta\,(\alpha\cos A-\beta\cos B)=0\dots\dots(10).$$

Another equation may be found in the same manner as (8),

considering only the tangents parallel to the lines $\alpha=0$, $\beta=0$. It is

$$c^2(\alpha^2-\beta^2)-4k(a^2-b^2)\alpha\beta+4kc(b\alpha-a\beta)\gamma=0\ldots(11).$$

It may be noticed that equations (9) and (10) each give four foci; this is no more than we might expect when we remember that to an ellipse or hyperbola there belong two imaginary foci satisfying the same analytical conditions as the real foci; in fact, in determining the eccentricity (e) of the general equation of the second degree in Cartesian coordinates, we obtain a quadratic in e^2, having one root positive and one negative, of which the former refers to the real foci, the latter to two imaginary foci; the two signs of e for each root to the different foci of each pair, the perpendiculars on the directrices being drawn in opposite directions. The existence of the four foci arises from the algebraical symmetry of the conic when referred to its principal axes.

The combination of (10) and (11) would however give six points; the reason of this is that from the method of investigation the equations would be satisfied if the angles ASC, BSC corresponding to one point were equal, and the angles $AS'C$, $BS'C$ corresponding to another supplemental; this cannot be true for the foci of a conic section.

Since the above mentioned symmetry does not exist in the case of the parabola, all trace of the imaginary foci ought to disappear, and the equations ought to give two points only, one at an infinite distance.

Thus since $P=0$ is the condition that (1) represents a parabola, the equations (8) give only a single point besides the point at an infinite distance given by the equation $H=0$.

If $\sqrt{(l\alpha)}\pm\sqrt{(m\beta)}\pm\sqrt{(n\gamma)}=0$ represent a parabola, $Q=0$, and equations (9) reduce to the simple form

$$\frac{l\alpha}{a^2}=\frac{m\beta}{b^2}=\frac{n\gamma}{c^2}.$$

The equations giving the focus of the parabola represented by $l\alpha^2+m\beta^2+n\gamma^2=0$, are

$$\frac{a\alpha}{m+n}=\frac{b\beta}{n+l}=\frac{c\gamma}{l+m},$$

and eliminating l, m, n by the condition

$$a^2mn+b^2nl+c^2lm=0,$$

we have $a^2(c\gamma+a\alpha-b\beta)(a\alpha+b\beta-c\gamma)$

$$+b^2(a\alpha+b\beta-c\gamma)(b\beta+c\gamma-a\alpha)+c^2(b\beta+c\gamma-a\alpha)(c\gamma+a\alpha-b\beta)=0.$$

Hence the following theorem: The locus of the focus of a parabola, such that with respect to it each angular point of a given triangle is the pole of the opposite side, is the circle circumscribing the triangle whose angular points are the middle points of the given triangle.

From equations (8) may be deduced an equation to determine the magnitudes of the semiaxes; for, since the equation connecting the axis major with the imaginary perpendiculars on any tangent from the two imaginary foci is the same as that connecting the axis minor with the perpendiculars from the real foci, each member of (8) will be equal to Pr^2, where r is one of the semiaxes; thus

$$P\alpha^2-\frac{dP}{da}H\alpha+\tfrac{1}{2}\frac{d^2P}{da^2}H^2=P\beta^2-\frac{dP}{db}H\beta+\tfrac{1}{2}\frac{d^2P}{db^2}H^2$$

$$=P\gamma^2-\frac{dP}{dc}H\gamma+\tfrac{1}{2}\frac{d^2P}{dc^2}H^2=Pr^2,$$

and eliminating α, β, γ by means of the relation

$$a\alpha+b\beta+c\gamma=H,$$

we obtain the equation giving the squares of the semiaxes

$$a\sqrt{(r^2+X)}\pm b\sqrt{(r^2+Y)}\pm c\sqrt{(r^2+Z)}=0,$$

where

$$\frac{X}{\left(\frac{dP}{da}\right)^2-2P\frac{d^2P}{da^2}}=\frac{Y}{\left(\frac{dP}{db}\right)^2-2P\frac{d^2P}{db^2}}=\frac{Z}{\left(\frac{dP}{dc}\right)^2-2P\frac{d^2P}{dc^2}}=\frac{H^2}{4P^2}.$$

Or, in terms of l, m, n...

$$\frac{X}{mc^2-2l'bc+nb^2}=\frac{Y}{na^2-2m'ca+lc^2}=\frac{Z}{lb^2-2n'ab+ma^2}$$

$$=-\frac{H^2}{P^2}\begin{vmatrix} l', & n, & m \\ n, & m', & l \\ m, & l, & n' \end{vmatrix}\ldots\ldots\ldots\ldots(12).$$

The above equation when rationalized is a quadratic in r^2

$$Er^4+2Fr^2+G=0\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

if $$E = a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2$$
$$= -(a+b+c)(a+b-c)(b+c-a)(c+a-b),$$
$$F = a^2X(a^2-b^2-c^2) + b^2Y(b^2-c^2-a^2) + c^2Z(c^2-a^2-b^2)$$
$$= -2abc(aX\cos A + bY\cos B + cZ\cos C),$$
$$G = a^4X^2 + b^4Y^2 + c^4Z^2 - 2b^2c^2YZ - 2c^2a^2ZX - 2a^2b^2XY.$$

Hence $$F^2 - EG = 4a^2b^2c^2\{a^2(X-Y)(X-Z)$$
$$+ b^2(Y-Z)(Y-X) + c^2(Z-X)(Z-Y)\},$$

the factor of this within the brackets by assuming

$$X - Y = p, \quad X - Z = q, \quad \text{and } Z - Y = p - q$$

becomes $$b^2p^2 + c^2q^2 + (a^2 - b^2 - c^2)pq,$$

and since $$a^2 > (b-c)^2, \text{ or } a^2 - b^2 - c^2 > -2bc,$$

this is greater than $b^2p^2 + c^2q^2 - 2bcpq$, which is always positive. Thus the equation (13) has always real roots in r^2.

Now if the equation (1) represent an ellipse both values of r^2 must be positive, if an hyperbola one must be positive and the other negative; the two values of r^2 will have the same or contrary signs according as $\frac{G}{E}$ is positive or negative, or since E is a negative quantity according as G is negative or positive.

Making use of the equations (12) the factor of G which may vary in sign will be of the eighth degree in a, b, c; involving a^4b^4, $a^4b^2c^2$, a^4b^3c, $a^2b^3c^3$, &c. and the coefficients of a^4b^4, a^4b^3c, ... vanish.

That of $a^4b^2c^2$ is $4(l'^2 - mn)$, and of $a^2b^3c^3$, $8(ll' - m'n')$. Hence the factor of G, which is of variable sign, is

$$(l'^2 - mn)a^2 + (m'^2 - nl)b^2 + (n'^2 - lm)c^2$$
$$+ 2(ll' - m'n')bc + 2(mm' - n'l')ca + 2(nn' - l'm')ab.$$

That is $(-P)$.

Thus we arrive at the conclusion that the curve represented by (1) is an ellipse, parabola, or hyperbola, according as the determinant P is $> = < 0$.

If the curve be a circle, then equation (13) must give equal roots for r^2; that is, we must have

$$F^2 - EG = 0.$$

This equation is satisfied if $X=Y=Z$, or

$$mc^2-2l'bc+nb^2=na^2-2m'ca+lc^2=lb^2-2n'ab+ma^2.$$

If the curve be a rectangular hyperbola, the two values of r^2 must be equal but of contrary signs; the condition for this is $F=0$, or

$$aX\cos A+bY\cos B+cZ\cos C=0,$$

$$bc(c\cos B+b\cos C)l+ca(a\cos C+c\cos A)m+ab(b\cos A+a\cos B)m$$
$$-2l'abc\cos A-2m'abc\cos B-2n'abc\cos C=0,$$

hence $l+m+n-2l'\cos A-2m'\cos B-2n'\cos C=0.$

July, 1861.

PASCAL'S THEOREM.

By H. W. Challis.

The intersections of opposite sides of a hexagon inscribed in a conic lie on one straight line.

Let P, Q (fig. 16) be intersections of two pairs of opposite sides. Suppose F to move along the conic, the sides CD, CB are cut each in a system of homographic points: such that C is the coincidence of two of the points, one in each system. Hence PQ, joining corresponding pairs of these, passes through a fixed point. But when F is at D, B, PQ respectively assumes the positions DE, AB; hence this fixed point must be the intersection of the last mentioned two straight lines. Whence the truth of the theorem.

November, 1861.

NOTE ON CERTAIN REMARKABLE PROPERTIES OF NUMBERS.

By JOHN BLISSARD, M.A.

THE following Properties of Numbers appear to be sufficiently remarkable, and I have arrived at them by a method which I think is new. I shall be greatly obliged if any reader of the *Journal* would kindly inform me whether they have ever been exhibited, or can be readily obtained by any known method.

Let $p_1, p_2, p_3 \dots p_n$ be the sums of the products of the n successive numbers $1, 2, 3 \dots n$, taken respectively $1, 2, 3 \dots n$ together; then (θ being perfectly arbitrary)

$$(1) \qquad \sin\theta\,(\cos\theta)^n\,p_n - \frac{n}{2}\sin 2\theta\,(\cos\theta)^{n-1}\,p_{n-1}$$

$$+\left(\frac{n}{2}\right)^2 \sin 3\theta\,(\cos\theta)^{n-2}\,p_{n-2} - \dots - \left(\frac{n}{2}\right)^n \sin(n+1)\,\theta = 0,$$

(n odd).

$$(2) \qquad (\cos\theta)^{n+1}\,p_n - \frac{n}{2}\cos 2\theta\,(\cos\theta)^{n-1}\,p_{n-1}$$

$$+\left(\frac{n}{2}\right)^2 \cos 3\theta\,(\cos\theta)^{n-2}\,p_{n-2} - \dots + \left(\frac{n}{2}\right)^n \cos(n+1)\,\theta = 0,$$

(n even).

Again, let $q_1, q_2, q_3 \dots q_n$ be the sums of the products of the n successive odd numbers $1, 3, 5 \dots 2n-1$; taken respectively $1, 2, 3 \dots n$ together; then (θ being arbitrary)

$$(3) \quad (\cos\theta)^n\,q_n - n\,(\cos\theta)^n\,q_{n-1} + n^2\cos 2\theta\,(\cos\theta)^{n-2}\,q_{n-2}$$

$$-\,n^3\cos 3\theta\,(\cos\theta)^{n-3}\,q_{n-3} + \dots - n^n\cos n\theta = 0,$$

(n odd).

$$(4) \qquad \sin\theta\,(\cos\theta)^{n-1}\,q_{n-1} - n\sin 2\theta\,(\cos\theta)^{n-2}\,q_{n-2}$$

$$+\,n^2\sin 3\theta\,(\cos\theta)^{n-3}\,q_{n-3} + \dots - n^{n-1}\sin n\theta = 0,$$

(n even).

Vicarage, Hampstead Norris,
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THEORY OF GENERIC EQUATIONS.

By JOHN BLISSARD, M.A.

(Continued from p. 75.)

CHAPTER III.

On Transcendents and Representative Notation.

ON the subject of Transcendents, the method and notation here used will, it is believed, furnish easy and elegant proof of various known formulæ, and also lead to the discovery of important new ones. I wish, however, before pursuing the subject further, in justification of my notation, to which I consider these novel results are especially due, to offer some remarks on Representative Notation.

§ (1) *On Representative Notation.*

The notation used in this theory, and which may not unaptly be termed 'Representative Notation,' consists in the adoption of a single conventionalism as follows:

Let $U_0, U_1, U_2 \dots U_n$ be any class or series either of quantities or functions, which are connected by any general law of relation, then U^n is held to be equivalent to, and may in development be replaced by U_n.

The above conventionalism appears to include the fundamental notation of all algebra, whether actual or symbolic, and although closely analogous to the received notation of the Calculus of Operations, is not identical with it, and may be advantageously distinguished from it principally in three respects.

(1) From avoiding the separation of symbols it possesses much greater simplicity and consequently in many cases much greater penetrating and generalizing power.

(2) It is of wholly unrestricted application and may not only be employed, usually with extensive generalization, *in every case* where the operative notation is admissible, but it also subjects to symbolic action numerous classes of quantities or functions which are properly beyond the reach of that notation.

(3) All the processes in which it is used are in *strict accordance* with the principles and ideas of common algebra,

from which the operative notation is in some respects a manifest departure.

These advantages are very considerable, and, if they can be established, will render the representative notation, I hope, not unworthy of consideration, as regards its alleged convenience and utility.

In confirmation of the above assertions, I venture to make the following observations:

1. I have employed the representative notation and the method grafted upon it in four different branches of analysis, viz. (1) in the development of functions, (2) in the evaluation of transcendents, (3) in the generalization of functions and of their expression by the introduction into their value of one or more arbitrary quantities, and (4) in the summation of series, and in all these branches I find that my notation enters with great ease into highly complicated processes, and produces novel results of a high degree of generality, in cases where the use of a divided notation, if not, as I believe, actually inadmissible, would at least prove intolerably cumbersome and embarrassing. Thus as regards developments, I obtain the general term in the expansion of numerous functions, such as

$$\left(\frac{2}{\varepsilon^\theta+1}\right)^m, \quad \left(\frac{\theta}{\varepsilon^\theta-1}\right)^m, \quad \frac{\cos m\theta}{(\cos\theta)^m}, \quad \left(\frac{\theta}{\sin\theta}\right)^m \cos m\theta, \quad \frac{\sin m\theta}{(\cos\theta)^m},$$

$$\left(\frac{\theta}{\sin\theta}\right)^m \sin m\theta, \quad \cos m\theta\,(\cos\theta)^m, \quad \cos m\theta\left(\frac{\sin\theta}{\theta}\right)^m, \text{ \&c.},$$

$$(\cos\theta)^m, \quad \frac{1}{(\cos\theta)^m}, \quad \left(\frac{\sin\theta}{\theta}\right)^m, \quad \left(\frac{\theta}{\sin\theta}\right)^m, \quad \varepsilon^{\varepsilon^\theta}, \quad \varepsilon^{\varepsilon^{\varepsilon^\theta}}, \text{ \&c.},$$

$$\left\{\frac{x}{\log(1+x)}\right\}^m, \quad \frac{x}{(1+x)^m-1}, \text{ \&c., \&c.}$$

Of course the general term in some of these developments is a complicated formula and requires a large amount of calculation, but still, as being the general term, not before exhibited, but now expressed either in terms of Bernoulli's numbers or the class of quantities called the 'differences of nothing,' or in some other finite and calculable form, it must be regarded, I conceive, as some advance in analytical science.

I have appended a large list of apparently new and highly general theorems and formulæ, in which the general term of some of these developments will be given.

2. The prominent idea of the operative notation, as its name imports, is that of the performance of successive known operations, and hence arose the desirableness of separating the symbol of operation from that of quantity. Thus $\frac{d^n fx}{dx^n}$ is an instance in which $\left(\frac{d}{dx}\right)^n$ is used to represent n successive operations performed on fx, and may as a symbol of operation be separated from the function on which it operates. But in the use of representative notation this restrictive idea of known operation need not be prominent nor even present to the mind at all. Thus I am able to use with great effect A^n, B^n, S^n, D^n, Γ^n, &c. for A_n, B_n, S_n, D_n, $\Gamma(n)$, &c. where B represents Bernoulli's numbers, A represents a class of numbers derived from them,

$$S_n = \frac{1}{1^n} + \frac{1}{2^n} + \text{\&c.},\ D_n = S_n - 1 = \frac{1}{2^n} + \frac{1}{3^n}, \text{\&c.},\ \Gamma(n) = 1.2\ldots(n-1).$$

In most of these cases and in many others, the idea of successive known operation does not exist. B^n is not derived from B_0 in at all the same way as $\frac{d^n fx}{dx^n}$ is derived by successive operation from fx. The difference between the two notations somewhat resembles that between differentiation and integration. It is always possible to differentiate, but we cannot always integrate. And so it is always possible to derive U_0 from U_n, as the representative notation may require, but we cannot always derive U_n from U_0, which is what is required by the operative notation.

Some theorems will be given which shew the effective manner in which Γ and D may be used as representative quantities.

3. In the list of theorems already referred to some will be found which satisfactorily shew that the representative notation can be very advantageously employed wherever the operative notation is now used, and it will appear, I think, that the former has a much wider scope than the latter, and is in fact inclusive of it, being applicable to classes of functions as well as to classes of quantities. When applied to functions, the two notations may have in some cases a close agreement, but still are not identical. Thus by operative notation $f(x+h)$ is expressed by $\varepsilon^{h\frac{d}{dx}}(fx)$, fx requiring to

be appended to every term in the development of $\varepsilon^{h\frac{d}{dx}}$. Now let $X_n = \frac{d^n fx}{dx^n}$; therefore $X_0 = fx$, then by representative notation $f(x+h) = X^0 \varepsilon^{Xh}$. Here X^0 may seem to perform the same office which fx does in the former expression, but in reality X^0 is strictly a factor, affecting as such, all the terms of ε^{Xh}. Thus

$$X^0\varepsilon^{Xh} = X^0\left(1 + Xh + \frac{X^2h^2}{1.2}, \&c.\right) = X^0 + Xh + \frac{X^2h^2}{1.2}, \&c.$$

$$= X_0 + X_1 h + \frac{X_2 h^2}{1.2}, \&c.$$

Here it will be seen that the ideas of common algebra are preserved in the one notation and lost in the other, X^0 being a factor in the one case and fx a mere appendage in the other.

4. The following list of general theorems and formulæ is selected out of numerous results obtained by the use of my method and notation. The demonstrations are for the most part omitted for the sake of brevity, but I may remark that, with the exception of a few which are somewhat difficult and elaborate, the rest are all obtained with remarkable conciseness and facility.

List of Selected Theorems.

I. Let B be the representative of Bernoulli's numbers, and let

$$(x+1)(x+2)\ldots(x+m-1) = x^{m-1} + q_1 x^{m-2} + q_2 x^{m-3} + \ldots + q_{m-1},$$

then

$$\left(\frac{\theta}{\varepsilon^\theta - 1}\right)^m = 1 - \frac{q_1\theta}{m-1} + \frac{q_2\theta^2}{(m-1)(m-2)} - \frac{q_3\theta^3}{(m-1)(m-2)(m-3)} + \ldots$$

$$+ (-1)^{m-1}\left\{\theta^{m-1} + \ldots + \frac{\theta^{m+n}}{\Gamma(m).\Gamma(n+1)}\left(\frac{B_{m+n}}{m+n} + q_1\frac{B_{m+n-1}}{m+n-1} + \ldots\right.\right.$$

$$\left.\left. + q_{m-1}\frac{B_{n+1}}{n+1}\right) + \ldots\right\}.$$

Ex. Let $m = 4$; therefore $q_1 = 6$, $q_2 = 11$, $q_3 = 6$, and by the above formula

$$\left(\frac{\theta}{\varepsilon^\theta - 1}\right)^4 = 1 - 2\theta + \frac{11\theta^2}{6} - \theta^3 + \frac{251}{720}\theta^4 - \&c.,$$

which is verified by actual evolution.

The proof of this formula is given in Chap. II. The coefficients of the above expansion are applicable in many ways. The following is an instance:

II. Let

$$\Delta_m \phi x = \phi(x+m) - \frac{m}{1}\phi(x+m-1) + \frac{m(m-1)}{1.2}\phi(x+m-2) - \&c.,$$

then

$$\frac{d^m}{dx^m}(\phi x) = \Delta_m \phi x - \frac{q_1}{m-1}\frac{d}{dx}(\Delta_m \phi x) + \frac{q_2}{(m-1)(m-2)}\frac{d^2}{dx^2}(\Delta_m \phi x) - \&c.$$

$$+(-1)^{m-1}\left\{\frac{d^{m-1}}{dx^{m-1}}(\Delta_m \phi x) + \ldots + \frac{1}{\Gamma(m)\,\Gamma(n+1)}\left(\frac{B_{m+n}}{m+n} + q_1\frac{B_{m+n-1}}{m+n-1} + \ldots\right.\right.$$

$$\left.\left. + q_{m-1}\frac{B_{n+1}}{n+1}\right)\frac{d^{m+n}}{dx^{m+n}}(\Delta_m \phi x) + \&c.\right\}.$$

Ex. Let $m=3$; therefore $q_1 = 3$, $q_2 = 2$; therefore

$$\frac{d^3(\phi x)}{dx^3} = \Delta_3 \phi x - \frac{3}{2}\frac{d}{dx}(\Delta_3 \phi x) + \frac{d^2}{dx^2}(\Delta_3 \phi x) - \frac{3}{8}\frac{d^3}{dx^3}(\Delta_3 \phi x)$$

$$+ \frac{19}{240}\frac{d^n}{dx^n}(\Delta_3 \phi x) - \&c.$$

Let $\phi(x) = x^7$; therefore

$$\Delta_3 \phi x = 210x^4 + 1260x^3 + 3150x^2 + 3780x + 1806,$$

and the formula, on verification, is found to hold.

III. Let

$$(x+1)(x+2)\ldots(x+n-1) = x^{n-1} + p_1 x^{n-2} + p_2 x^{n-3} + \ldots + p_{n-1},$$

and let $\left\{\dfrac{x}{\log(1+x)}\right\}^m = 1 + P_1 x + P_2 x^2 + \ldots + P_n x^n$, &c.,

then $P_n = \dfrac{1}{1.2\ldots n}\left\{\dfrac{\Delta^m 0^{n+m}}{(n+1)(n+2)\ldots(n+m)} - p_1\dfrac{\Delta^m 0^{n+m-1}}{n(n+1)\ldots(n+m-1)}\right.$

$$\left. + p_2\frac{\Delta^m 0^{n+m-2}}{(n-1)\,n\ldots(n+m-2)} - \&c. + (-1)^{n-1}p_{n-1}\frac{\Delta^m 0^{m+1}}{2.3\ldots(m+1)}\right\}.$$

Ex. Let $m=1$, then since $\Delta' 0^n = 1$, if

$$\frac{x}{\log(1+x)} = 1 + P_1 x + \ldots + P_n x^n, \&c.,$$

$$P_n = \frac{1}{1.2\ldots n}\left(\frac{1}{n+1} - \frac{p_1}{n} + \frac{p_2}{n-1} - \&c. + (-1)^{n-1}\frac{p_{n-1}}{2}\right),$$

which is easily verified.

IV. Let

$$\left(\frac{\sin\theta}{\theta}\right)^m = 1 - P_2\theta^2 + P_4\theta^4 - \&c. + (-1)^n P_{2n}\theta^{2n}\ \&c.,$$

then $P_{2n} = \dfrac{2^{2n}}{1.2.3\ldots(m+2n)}\left\{\Delta^m 0^{m+2n} - \dfrac{m+2n}{1}\cdot\dfrac{m}{2}.\Delta^m 0^{m+2n-1}\right.$

$$\left. + \frac{(m+2n).(m+2n-1)}{1.2}\cdot\left(\frac{m}{2}\right)^2 \Delta^m 0^{m+2n-2} - \&c.\right\}.$$

Ex. Let $m=3$, $n=2$, then if

$$\left(\frac{\sin\theta}{\theta}\right)^3 = 1 - P_2\theta^2 + P_4\theta^4 - \&c.,$$

$$P_4 = \frac{2^4}{1.2\ldots7}\{\Delta^3 0^7 - 7\left(\tfrac{3}{2}\right)\Delta^3 0^6 + 21\left(\tfrac{3}{2}\right)^2.\Delta^3 0^5$$

$$- 35\left(\tfrac{3}{2}\right)^3.\Delta^3 0^4 + 35\left(\tfrac{3}{2}\right)^4\Delta^3 0^3\}$$

$$= \tfrac{1}{315}\{1806 - 7\left(\tfrac{3}{2}\right)540 + 21\left(\tfrac{9}{4}\right)150 - 35\left(\tfrac{27}{8}\right)36 + 35\left(\tfrac{81}{16}\right)6\} = \tfrac{13}{120},$$

which, by actual evolution of $\left(1 - \dfrac{\theta^2}{6} + \dfrac{\theta^4}{120} - \&c.\right)^3$, is found to be the case.

Cognate with the above formula is the following:

V. $\Delta^m 0^{m+n} - \dfrac{m+n}{1}\cdot\left(\dfrac{m}{2}\right)\Delta^m 0^{m+n-1}$

$$+ \frac{(m+n).(m+n-1)}{1.2}\left(\frac{m}{2}\right)^2 \Delta^m 0^{m+n-2} - \&c. = 0,\ (n \text{ odd}).$$

Ex. $(m=3,\ n=3)$,

$$\Delta^3 0^6 - 6\left(\tfrac{3}{2}\right)\Delta^3 0^5 + 15\left(\tfrac{3}{2}\right)^2\Delta^3 0^4 - 20\left(\tfrac{3}{2}\right)^3\Delta^3 0^3 = 0,$$

i.e., $\quad 540 - 6\left(\tfrac{3}{2}\right)150 + 15\left(\tfrac{9}{4}\right)36 - 20\left(\tfrac{27}{8}\right)6 = 0,$

which is the case.

NOTE. The demonstrations of (IV.) and (V.) will not take up much room, and may therefore here be given.

From the formation of $\Delta^m 0^n$,

$$(\varepsilon^\theta - 1)^m = \frac{\Delta^m 0^m(\theta)^m}{1.2\ldots m} + \frac{\Delta^m 0^{m+1}(\theta)^{m+1}}{1.2\ldots(m+1)} + \frac{\Delta^m 0^{m+2}(\theta)^{m+2}}{1.2\ldots(m+2)} + \&c.,$$

assume $U_n = \Delta^m 0^n$, then by representative notation,

$$\varepsilon^{U\theta} - 1 = (\varepsilon^\theta - 1)^m;$$

therefore $\varepsilon^{(2U-m)\theta\sqrt{(-1)}} - 1 = (\varepsilon^{\theta\sqrt{(-1)}} - \varepsilon^{-\theta\sqrt{(-1)}})^m = 2^m(-1)^{\frac{1}{2}m}(\sin\theta)^m$

$$= 2^m(-1)^{\frac{1}{2}m}\theta^m\{1 - P_2\theta^2 + P_4\theta^4 - \&c. + (-1)^n P_{2n}\theta^{2n}\ \&c.\};$$

therefore equating coefficients of θ^{m+2n},

$$\frac{(2U-m)^{m+2n}}{1.2.3\ldots(m+2n)}=2^m P_{2n},$$

i.e.,
$$P_{2n}=\frac{2^{2n}}{1.2\ldots(m+2n)}\left(U-\frac{m}{2}\right)^{m+2n},$$

which being expanded gives (IV.). Also, (n being odd),

$$\frac{(2U-m)^{m+n}}{1.2\ldots(m+n)}=0;$$

therefore
$$\left(U-\frac{m}{2}\right)^{m+n}=0,$$

which being expanded gives (V.).

It may be observed here that the application of representative notation to the $\Delta^m 0^n$ numbers yields a large supply of general formulæ, some of which are remarkable.

VI. The following is an instance.

Let $b_1, b_2, b_3, \ldots b_{m-1}$ be the sums of the products of the $(m-1)$ quantities $1^p, 2^p, 3^p, \ldots (m-1)^p$, taken $1, 2, 3, \ldots (m-1)$ together, then

$$\Delta^m 0^{pm} - b_1\Delta^m 0^{p(m-1)} + b_2\Delta^m 0^{p(m-2)} - \&c.$$
$$+(-1)^{m-1} b_{m-1}\Delta^m 0^p = m^p(m^p-1^p)(m^p-2^p)\ldots\{m^p-(m-1)^p\}.$$

Ex. Let $m=3$, $p=3$; therefore

$$b_1 = 1^3+2^3=9,\quad b_2=1^3.2^3=8,$$

and $\Delta^3 0^9 - 9\Delta^3 0^6 + 8\Delta^3 0^3 = 3^3(3^3-1^3)(3^3-2^3) = 27.26.19,$

i.e.,
$$18150-9.540+8.6=13338,$$

which is the case.

VII. The following results appear useful and worthy of mention:

1st. $\Delta^m 0^{m+n} = (a_1 m^n + a_2 m^{n-1} + \ldots + a_n m)\dfrac{\Gamma(m+n+1)}{\Gamma(2n+1)}$,

when $a_1, a_2, \ldots a_n$ are found by developing $(\epsilon^\theta-1)^m$. Thus

(1) $\Delta^m 0^{m+1} = m\dfrac{\Gamma(m+2)}{\Gamma 3}$,

(2) $\Delta^m 0^{m+2} = (3m^2+m)\dfrac{\Gamma(m+3)}{\Gamma 5}$,

(3) $\Delta^m 0^{m+3} = 15\,(m^3 + m^2)\,\frac{\Gamma(m+4)}{\Gamma 7}$,

(4) $\Delta^m 0^{m+4} = (105m^4 + 210m^3 + 35m^2 - 14m)\,\frac{\Gamma(m+5)}{\Gamma 11}$,

(5) $\Delta^m 0^{m+5} = 315m^2\,(m+1)\,(3m^2 + 7m - 2)\,\frac{\Gamma(m+6)}{\Gamma 13}$ &c., &c.

The formula holds for any value of m.

2nd. $$\Delta^{-m}0^n = \frac{(-1)^{m-1}}{1.2\ldots(m-1)}\left(\frac{B_{n+m}}{n+m} + q_1\,\frac{B_{n+m-1}}{n+m-1} + q_2\,\frac{B_{n+m-2}}{n+m-2} + \ldots + q_{m-1}\,\frac{B_{n+1}}{n+1}\right) \text{ (see I.)}.$$

3rd. $$q_n = (-1)^n\,(a_1 m^n - a_2 m^{n-1} + a_3 m^{n-2} - \text{\&c.})\,\frac{\Gamma(m)}{\Gamma(m-n).\Gamma(2n+1)}.$$

Hence $q_1 = \frac{m.(m-1)}{1.2}$, $q_2 = (3m-1)\,\frac{m.(m-1).(m-2)}{1.2.3.4}$,

$$q_3 = 15\,(m^2 - m)\,\frac{m.(m-1).(m-2).(m-3)}{1.2.3.4.5.6} \text{ \&c.*}$$

VIII. The following formula has been proved (Chap. II.).

Let $$\left(\frac{2}{\varepsilon^\theta + 1}\right)^m = 1 + U_1\theta + U_2\,\frac{\theta^2}{1.2} + \ldots + \frac{U_n\theta^n}{1.2\ldots n} \text{ \&c.},$$

then U_n has two expressions:

(1) $$U_n = (-1)^n\left[m^n - \frac{m}{1}\left\{\frac{(m+1)^n - m^n}{2}\right\} + \frac{m.(m+1)}{1.2}\left\{\frac{(m+2)^n - 2.(m+1)^n + m^n}{2^2}\right\} - \text{\&c.}\right].$$

(2) $$U_n = -\frac{2^m}{1.2\ldots(m-1)}\left\{(2^{n+m}-1)\,\frac{B_{n+m}}{n+m} + q_1\,(2^{n+m-1}-1)\,\frac{B_{n+m-1}}{n+m-1} + \ldots + q_{m-1}\,(2^{n+1}-1)\,\frac{B_{n+1}}{n+1}\right\}.$$

* These results ought to have been given as Corollaries to Theorem (I.).

IX. The following formula is analogous to the preceding one.

Let $\sigma_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \text{\&c. which} = \left(1 - \frac{1}{2^{n-1}}\right) S_n \, (n > 1)$;

therefore $\sigma_1 = \frac{1}{1} - \frac{1}{2} + \text{\&c.} = \log 2$, $\sigma_0 = 1 - 1 + 1 - \text{\&c.} = \frac{1}{2}$,

$$\sigma_{-1} = 1 - 2 + 3 - \text{\&c.} = \frac{1}{(1+1)^2} = \frac{1}{4}, \text{ \&c.,}$$

then
$$\frac{1}{m^n} + \frac{m}{1}\left\{\frac{1}{m^n} - \frac{1}{(m+1)^n}\right\} \cdot \frac{1}{2}$$
$$+ \frac{m(m+1)}{1.2}\left\{\frac{1}{m^n} - \frac{2}{(m+1)^n} + \frac{1}{(m+2)^n}\right\} \cdot \frac{1}{2^2} - \text{\&c.}$$
$$= (-1)^{m-1} \frac{2^m}{1.2\ldots(m-1)} \{\sigma_{n-m+1} - q_1\sigma_{n-m+2}$$
$$+ q_2\sigma_{n-m+3} - \ldots + (-1)^{m-1} q_{m-1}\sigma_n\},$$

where, as before,

$$(x+1)(x+2)\ldots(x+m-1) = x^{m-1} + q_1 x^{m-2} + q_2 x^{m-3} + \ldots + q_{m-1}.$$

Ex. 1. Let $m = 3$, $n = 1$, then

$$\frac{1}{3} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2^2}, \text{ \&c.} = 4(\sigma_{-1} - 3\sigma_0 + 2\sigma_1) = 8 \log 2 - 5.$$

Ex. 2. Let $m = r$, then

$$1 + \frac{1}{2}\left(1 - \frac{1}{2^n}\right) + \frac{1}{2^2}\left(1 - \frac{2}{2^n} + \frac{1}{3^n}\right) + \text{\&c.} = 2\sigma_n = \left(2 - \frac{1}{2^{n-2}}\right) S_n,$$

if $\quad n = 2, \; 1 + \frac{3}{8} + \frac{11}{72} + \frac{1}{24} + \text{\&c.} = S_2 = \frac{\pi^2}{6}.$

X. The following theorem may be regarded as a generalization of the multinomial expansion, since the latter is derived from it as a particular case.

Let ϕx be a developable function and

$$= U_0 + U_1 x + \ldots + U_n x^n, \text{ \&c.,}$$

and let fx be such that $\frac{d^n}{dx^n}(fx)$ can be expressed as a function of x, then $f(\phi x)$ is also a developable function whose general term is given by the following formula:

Let $\quad \frac{d^n}{dx^n}(fx)\,(x = U_0) = V_n,$

and let $\quad f(\phi x) = C_0 + C_1 x + \ldots + C_n x^n + \text{\&c.},$

then $$C_n = \Sigma\left\{\frac{U_p^{\alpha}.U_q^{\beta}.U_r^{\gamma}}{\Gamma(\alpha+1).\Gamma(\beta+1).\Gamma(\gamma+1)\ldots}\times V_k\right\},$$

where $$\alpha+\beta+\gamma, \text{ \&c.} = k,$$

and $$p\alpha+q\beta+r\gamma, \text{ \&c.} = n.$$

Hence $$C_n = \frac{U_1^n}{\Gamma(n+1)}.V_n + \frac{U_1^{n-2}U_2}{\Gamma(n-1)}.V_{n-1}$$

$$+\left\{\frac{U_1^{n-4}U_2^2}{\Gamma(3)\,\Gamma(n-3)}+\frac{U_1^{n-3}U_3}{\Gamma(n-2)}\right\}V_{n-2}+\left\{\frac{U_1^{n-4}U_4}{\Gamma(n-3)}+\frac{U_1^{n-5}U_2U_3}{\Gamma(n-4)}\right\}V_{n-3}+\ldots.$$

Here the formation of terms is easy, observing that every term in the factor of V_k is to consist of k factors, the sum of whose subindices $= n$.

Ex. 1. Let $fx = x^m$, then

$$V_k = \frac{d^k}{dx^k}(x^m)(x = U_0) = m(m-1)\ldots(m-k+1)\,U_0^{m-k}$$
$$= \frac{\Gamma(m+1)}{\Gamma(m-k+1)}\,U_0^{m-k}.$$

Hence if $(U_0+U_1x+\ldots+U_nx^n, \text{ \&c.})^m = C_0+C_1x+\ldots+C_nx^n$, &c.,

$$C_n = \Gamma(m+1)\,\Sigma\left\{\frac{U_p^{\alpha}.U_q^{\beta}.U_r^{\gamma}}{\Gamma(\alpha+1).\Gamma(\beta+1).\Gamma(\gamma+1)\ldots}\times\frac{U_0^{m-k}}{\Gamma(m-k+1)}\right\},$$

which is the multinomial expansion.

Ex. 2. Let $\phi x = \sin x$, $fx = \varepsilon^x$; therefore

$$U_0 = 0,\ U_1 = 1,\ U_2 = 0,\ U_3 = -\frac{1}{1.2.3}, \text{ \&c.},$$

and $$V_k = \frac{d^k}{dx^k}(\varepsilon^x)_{x=0} = 1.$$

Hence if $\varepsilon^{\sin x} = 1 + C_1x + C_2x^2 + \ldots + C_nx^n$, &c.,

let $n = 3$; therefore

$$C_3 = \frac{U_1^3}{\Gamma(4)} + \frac{U_3}{\Gamma(1)} = \frac{1}{6} - \frac{1}{6} = 0 \text{ which is the case,}$$

n = 5; therefore

$$C_5 = \frac{U_1^5}{\Gamma(6)} + \frac{U_1^2U_3}{\Gamma(3)} + \frac{U_5}{\Gamma(6)} = \frac{1}{120} - \frac{1}{12} + \frac{1}{120} = -\frac{1}{15}.$$

The two following theorems illustrate the use made of Γ and D as representative quantities

XI. Let $\Gamma^{-n} = \frac{1}{\Gamma(n)}$; therefore $\varepsilon^\theta = \frac{1}{\Gamma} + \frac{\theta}{\Gamma^2} + \frac{\theta^2}{\Gamma^3} + \ldots = \frac{1}{\Gamma - \theta}$.

Operating on this equation, we obtain

$$\frac{\varepsilon^\theta}{\Gamma^m}\left(1+\frac{\theta}{\Gamma}\right)^n = \frac{\Gamma^n}{(\Gamma-\theta)^{m+n}}.$$

$$\text{Hence } \varepsilon^\theta = \frac{1 + \frac{m+n}{m} \cdot \frac{\theta}{1} + \frac{(m+n)(m+n+1)}{m(m+1)} \cdot \frac{\theta^2}{1.2}, \&c.}{1 + \frac{n}{m} \cdot \frac{\theta}{1} + \frac{n(n-1)}{m(m+1)} \cdot \frac{\theta^2}{1.2}, \&c.},$$

where m and n are perfectly arbitrary.

If we put $-n$ for n and express

$$1 - \frac{n(n+1)}{m(m+1)} \cdot \frac{\theta^2}{1.2} + \frac{n\ldots(n+3)}{m\ldots(m+3)} \cdot \frac{\theta^4}{1.2.3.4} - \&c.,$$

which follows a cosine analogy by $C(n, \theta)$, and

$$\frac{n}{m} \cdot \frac{\theta}{1} - \frac{n(n+1).(n+2)}{m(m+1).(m+2)} \cdot \frac{\theta^3}{1.2.3}, \&c.,$$

which follows a sine analogy by $S(n, \theta)$, we easily derive from the above the following remarkably symmetrical formulæ

$$\cos\theta = \frac{C(n,\theta)\, C\{(m-n), \theta\} - S(n,\theta)\, S\{(m-n), \theta\}}{S^2(n,\theta) + C^2(n,\theta)},$$

$$\sin\theta = \frac{S(n,\theta)\, C\{(m-n), \theta\} + S\{(m+n), \theta\}\, C(n,\theta)}{S^2(n,\theta) + C^2(n,\theta)}.$$

Hence $S^2(n, \theta) + C^2(n, \theta) = S^2\{(m-n), \theta\} + C^2\{(m-n), \theta\}$, from which is derived the following property of numbers, that if $S^2(n, \theta) + C^2(n, \theta)$ be expanded according to the powers of θ, each coefficient must be a symmetrical function of n and $(m-n)$, which is found to be the case. Thus let

$$C^2(n, \theta) + S^2(n, \theta) = 1 + P_2\theta^2 + P_4\theta^4, \&c.,$$

$$P_2 = \left(\frac{n}{m}\right)^2 - \frac{n(n+1)}{m(m+1)} = \frac{n(n-m)}{m(m+1)},$$

$$P_4 = \frac{1}{4}\left\{\frac{n(n+1)}{m(m+1)}\right\}^2 + \frac{1}{12}\,\frac{n(n+1).(n+2).(n+3)}{m(m+1).(m+2).(m+3)} - \frac{1}{3}\,\frac{n^2(n+1).(n+2)}{m^2(m+1).(m+2)},$$

i.e. $$P_4 = \frac{1}{2}\,\frac{n(n+1).(n-m).(n-m+1)}{m^2(m+1)^2.(m+2).(m+3)}, \&c.$$

The above formulæ are instances of the generalization of the expression of a function by the introduction into its value of the arbitrary quantities m and n, and in the production of these results, to have employed a symbol of operation, as distinct from that of quantity, would have been worse than useless and altogether unmeaning.

XII. Let $D^n = D_n = \frac{1}{2^n} + \frac{1}{3^n}$, &c. $= S_n - 1$,

then from the well known equation

$$\frac{n}{1} . D_{n+1} - \frac{n(n+1)}{1.2} . D_{n+2} + \frac{n(n+1).(n+2)}{1.2.3} . D_{n+3} - \&c. = \frac{1}{2^n} \; (n > 0),$$

which, expressed by representative notation, becomes

$$D^n - \left(\frac{D}{1+D}\right)^n = \frac{1}{2^n},$$

may be derived as a first generalization, n being perfectly arbitrary,

$$\left.\begin{array}{l} \{n(n+1) - n(n-1)\} \dfrac{D_2}{1.2} \\ + \{n(n+1).(n+2).(n+3) - n(n-1).(n-2).(n-3)\} \dfrac{D_4}{1.2.3.4}, \&c. = \dfrac{2^{2n}-1}{2^{n+1}} \\ \{n(n+1).(n+2) - n(n-1).(n-2)\} \dfrac{D_3}{1.2.3} \\ + \{n(n+1)\ldots(n+4) - n(n-1)\ldots(n-4)\} \dfrac{D_5}{1.2.3.4.5}, \&c. = \dfrac{(2^n-1)^2}{2^{n+1}} \end{array}\right\} \ldots\ldots\ldots\ldots(1),$$

$n = 0$ gives $\frac{D_2}{1} + \frac{D_4}{2} + \frac{D_6}{3}$, &c. $= \log 2$ (a well known formula)

$$\text{and } \left(1 + \frac{1}{2}\right)\frac{D_3}{3} + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)\frac{D_5}{5} + \left(1 + \frac{1}{2} + \ldots + \frac{1}{6}\right)\frac{D_7}{7}, \&c. = \frac{(\log 2)^2}{4},$$

$$n = \frac{1}{2} \text{ gives } \frac{1}{2} D_2 + \frac{1.3.5}{2.4.6} D_4 + \frac{1.3.5.7.9}{2.4.6.8.10} D_6, \&c. = \frac{1}{2\sqrt{(2)}},$$

$$\frac{1.3}{2.4} . \frac{2}{3} D_3 + \frac{1.3.5.7}{2.4.6.8} . \frac{4}{5} D_5, \&c. = \frac{3}{2\sqrt{(2)}} - 1,$$

$$n=\sqrt{(-1)} \text{ gives } \sin(\log 2)=D_2-\frac{D_4}{4}-\frac{11D_6}{36}-\&c.,$$

$$1-\cos(\log 2)=D_3+\frac{2D_5}{3}+\frac{5D_7}{12}+\&c.$$

By a higher generalization, we obtain

$$\left.\begin{array}{l} mD_{m+1}+\{(m+n).(m+n+1).(m+n+2)-n(n-1).(n-2)\}\dfrac{D_{m+3}}{1.2.3}, \&c. \\ \qquad\qquad =2^{n-1}+\dfrac{1}{2^{m+n-1}} \\ \{(m+n).(m+n+1)-n(n-1)\}\dfrac{D_{m+2}}{1.2} \\ +\{(m+n)\ldots(m+n+3)-n\ldots(n-3)\}\dfrac{D_{m+4}}{1.2.3.4}, \&c.=2^{n-1}-\dfrac{1}{2^{m+n-1}} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(2).$$

Very remarkable results may be obtained from these formulæ (by the substitution of negative integral values for m) which cannot here conveniently be specified.

The proof of the above theorems is short and is as follows:

Since $$D^n-\left(\frac{D}{1+D}\right)^n=\frac{1}{2^n};$$

therefore $$f(h+D\theta)-f\left(h+\frac{D\theta}{1+D}\right)=f\left(h+\frac{\theta}{2}\right)-f(h),$$

let $$h=1,\ \theta=-1, \text{ and } fh=h^{-n};$$

therefore $$(1-D)^{-n}-(1+D)^n=2^n-1;$$

therefore, putting $-n$ for n,

$$(1-D)^n-(1+D)^{-n}=2^{-n}-1.$$

Then, by subtracting and adding these equations, we obtain, on expansion, the equations (1).

Similarly

$$D^m f(h+D\theta)-\left(\frac{D}{1+D}\right)^m f\left(h+\frac{D\theta}{1+D}\right)=\frac{1}{2^m}f\left(h+\frac{\theta}{2}\right) m>0,$$

hence ($h=1$, $\theta=-1$, $fh=h^n$) $D^m(1-D)^n-D^m(1+D)^{-(m+n)}=\dfrac{1}{2^{m+n}}$,

putting $-(m+n)$ for n,

$$D^m(1-D)^{-(m+n)}-D^m(1+D)^n=2^n,$$

whence by adding, subtracting, and expanding, we get equations (2).

XIII. The operative notation has been used with great effect in the solution of differential equations. The following theorem will shew that the representative notation can here also be advantageously employed.

Let $$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n = fx.$$

This, by operative notation, becomes

$$\left\{\psi\left(\frac{d}{dx}\right)\right\} y = fx,$$

where $$\psi\left(\frac{d}{dx}\right) = \left(\frac{d}{dx}\right)^n + a_1\left(\frac{d}{dx}\right)^{n-1} + \ldots + a_n,$$

and the solution obtained is

$$y = \left\{\frac{1}{\psi\left(\frac{d}{dx}\right)}\right\} fx.$$

By use of representative notation the following extension is made with great facility, viz.

$$\phi\left(\frac{d}{dx}\right)\Bigg\} y = \left\{\frac{\phi\left(\frac{d}{dx}\right)}{\psi\,\frac{d}{dx}}\right\} fx.$$

Ex. Let $$\left\{\cos\left(\frac{d}{dx}\right)\right\} y = fx,$$

and let $$\phi\left(\frac{d}{dx}\right) = \sin\left(\frac{d}{dx}\right),$$

then $$\left\{\sin\left(\frac{d}{dx}\right)\right\} y = \left\{\tan\left(\frac{d}{dx}\right)\right\} fx.$$

Let $y = x^4$; therefore

$$fx = \left\{\cos\left(\frac{d}{dx}\right)\right\} x^4 = x^4 - 6x^2 + 1,$$

and $$\left\{\sin\left(\frac{d}{dx}\right)\right\} x^4 = 4x^3 - 4x.$$

Hence $\left\{\tan\left(\frac{d}{dx}\right)\right\} (x^4 - 4x^2 + 1)$ ought to $= 4x^3 - 4x$, which is the case.

XIV. In Gregory's Examples, *Differential and Integral Calculus*, p. 247, the following theorem, obtained by means of the operative notation, is given as an extension of one discovered by Mr. Murphy, viz.

$$fx = f(x-nh) + nhf'\{x-(n+1)h\} + n(n+2)\frac{h^2}{1.2}f''\{x-(n+2)h\}$$

$$+ n(n+3)\frac{h^3}{1.2.3}f'''\{x-(n+3)h\} + \text{\&c.}$$

Of this theorem I obtain, by use of representative notation, a highly elaborate and extensive generalization, which does not seem capable of being arrived at in any other way.

Let f_1, f_2, f_3, &c. be used instead of f', f'', f''', &c., and let P, Q be any functions of h, each containing h as a factor; also let $P_n = \frac{d^nP}{dh^n}$ and $Q_n = \frac{d^nQ}{dh^n}$, then

$$\frac{fx - f(x-Q)}{Q_1} + \int \frac{Q_2}{Q_1^2}\{fx - f(x-Q)\}\,dh = hf_1\{x-(P+Q)\}$$

$$+ \frac{h^2}{1.2}(2P_1+Q_1)f_2\{x-(2P+Q)\} + \frac{h^3}{1.2.3}\left[-(3P_2+Q_2)f_2\{x-(3P+Q)\}\right.$$

$$\left.+(2P_1+Q_1)^2f_3\{x-(3P+Q)\}\right] + \frac{h^4}{1.2.3.4}\left[(4P_3+Q_3)f_2\{x-(4P+Q)\}\right.$$

$$\left.-3(4P_1+Q_1)(4P_2+Q_2)f_3\{x-(4P+Q)\} + (4P_1+Q_1)^3f_4\{x-(4P+Q)\}\right]$$

$$+ \frac{h^5}{1.2.3.4.5}\left[-(5P_4+Q_4)f_2\{x-(5P+Q)\}\right.$$

$$+ \{3(5P_2+Q_2)^2 + 4(5P_1+Q_1)(5P_3+Q_3)\}f_3\{x-(5P+Q)\}$$

$$\left.-6(5P_1+Q_1)^2(5P_2+Q_2)f_4\{x-(5P+Q)\} + (5P_1+Q_1)^4f_5\{x-(5P+Q)\}\right]$$

$$+ \text{\&c., \&c.}$$

The law of formation of terms in the above general formula is as follows: The general term in the series is expressed by

$$\frac{h^n}{1.2\ldots n}\Sigma\left[(-1)^k(nP_p+Q_p)^\alpha.(nP_q+Q_q)^\beta.(nP_r+Q_r)^\gamma\ldots\right.$$

$$\times f_{k+1}\{x-(nP+Q)\}$$

$$\left.\times\frac{\Gamma(n)}{\Gamma(p+1)^\alpha.\Gamma(q+1)^\beta.\Gamma(r+1)^\gamma\ldots\Gamma(\alpha+1).\Gamma(\beta+1).\Gamma(\gamma+1)}\right],$$

where $\alpha+\beta+\gamma+\ldots=k,\ p\alpha+q\beta+r\gamma+\ldots=n-1.$

Ex. 1. Let $P=h,\ Q=nh;$

therefore $P_1=1,\ P_2=0$, &c., $Q_1=n,\ Q_2=0$, &c.,

$$(2P_1+Q_1)=n+2,\ 3P_1+Q_1=n+3, \text{ \&c.};$$

therefore

$$\frac{fx-f(x-nh)}{n}=hf_1\{x-(n+1)h\}+\frac{h^2}{1.2}(n+2)f_2\{x-(n+2)h\}+\text{\&c.},$$

which is the formula given by Gregory.

Ex. 2. Let $P=h,\ Q=h^2;$

therefore $P_1=1,\ P_2=0$, &c., $Q_1=2h,\ Q_2=2,$

also let $fx=x^3$; therefore

$$\frac{x^3-(x-h^2)^3}{2h}+\int\frac{x^3-(x-h^2)^3}{2h^2}\,dh=3h\{x-(h+h^2)\}^2$$

$$+\frac{h^2}{2}(2+2h)6\{x-(2h+h^2)\}+\frac{h^3}{6}[-12\{x-(3h+h^2)\}+(3+2h)^2 6]$$

$$+\frac{h^4}{24}\{-3(4+2h)12\}+\frac{h^5}{120}\cdot(3.2^2.6),$$

which on expansion is found to be the case.

XV. With regard to the summations of infinite series, a large number of results can be obtained, of which the following are instances:

$$\Sigma_{+\infty}^{-\infty}\frac{1}{(mx+p)^n}=\frac{(-1)^{\frac{1}{2}(n-1)}}{\Gamma(n)}\left(\frac{2\pi}{m}\right)^n\left\{\frac{1}{2}\frac{\cos\frac{\pi p}{m}}{\sin\frac{\pi p}{m}}-\frac{2^{n-1}-1}{2^2}\cdot\frac{\sin\frac{2\pi p}{m}}{\sin\frac{2\pi p}{m}}\right.$$

$$\left.-\frac{3^{n-1}-2.2^{n-1}+1}{2^3}\cdot\frac{\cos\frac{3\pi p}{m}}{\sin^3\frac{\pi p}{m}}+\frac{4^{n-1}-3.3^{n-1}+3.2^{n-1}-1}{2^4}\cdot\frac{\sin\frac{4\pi p}{m}}{\sin^4\frac{\pi p}{m}}+\text{\&c.}\right\}$$

(n odd)............................(1),

$$\text{and} = \frac{(-1)^{\frac{1}{2}n+1}}{\Gamma(n)}\left(\frac{2\pi}{m}\right)^n\left\{\frac{1}{2} + \frac{2^{n-1}-1}{2^2}\cdot\frac{\cos\frac{2\pi p}{m}}{\sin^2\frac{\pi p}{m}}\right.$$

$$-\frac{3^{n-1}-2.2^{n-1}+1}{2^3}\cdot\frac{\sin\frac{3\pi p}{m}}{\sin^3\frac{\pi p}{m}}$$

$$\left.-\frac{4^{n-1}-3.3^{n-1}+3.2^{n-1}-1}{2^4}\cdot\frac{\cos\frac{4\pi p}{m}}{\sin^4\frac{\pi p}{m}}+\&c.\right\}^* \ (n \text{ even})\ldots.(2).$$

Ex. 1. In (1), let $m=3$, $p=1$, then

$$\Sigma_{+\infty}^{-\infty}\frac{1}{(3x+1)^n} = \left(\frac{1}{1^n}+\frac{1}{4^n}+\frac{1}{7^n}, \&c.\right) - \left(\frac{1}{2^n}+\frac{1}{5^n}+\frac{1}{8^n}, \&c.\right)$$

$$= \frac{(-1)^{\frac{1}{2}n-1}}{\Gamma n}\left(\frac{2\pi}{3}\right)^n\left\{\frac{1}{2}\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} - \frac{2^{n-1}-1}{2^2}\cdot\frac{\sin\frac{2\pi}{3}}{\sin^2\frac{\pi}{3}} - \&c.\right\}.$$

* If the $\Delta^m 1^n$ notation be used, the above may be more briefly expressed as follows: n being odd,

$$\Sigma_{+\infty}^{-\infty}\frac{1}{(mx+p)^n} = \frac{(-1)^{\frac{1}{2}(n-1)}}{\Gamma(n)}\left(\frac{2\pi}{m}\right)^n\left\{\frac{\Delta^0 1^{n-1}}{2}\cdot\frac{\cos\frac{\pi p}{m}}{\sin\frac{\pi p}{m}}\right.$$

$$\left.-\frac{\Delta^1 1^{n-1}}{2^2}\cdot\frac{\sin\frac{2\pi p}{m}}{\sin^2\frac{\pi p}{m}} - \frac{\Delta^2 1^{n-1}}{2^3}\cdot\frac{\cos\frac{3\pi p}{m}}{\sin^3\frac{\pi p}{m}} + \&c.\right\},$$

n being even,

$$\Sigma_{+\infty}^{-\infty}\frac{1}{(mx+p)^n} = \frac{(-1)^{\frac{1}{2}n+1}}{\Gamma(n)}\left(\frac{2\pi}{m}\right)^n\left\{\frac{\Delta^0 1^{n-1}}{2} + \frac{\Delta^1 1^{n-1}}{2^2}\cdot\frac{\cos\frac{2\pi p}{m}}{\sin^2\frac{\pi p}{m}}\right.$$

$$\left.-\frac{\Delta^2 1^{n-1}}{2^3}\cdot\frac{\sin\frac{3\pi p}{m}}{\sin^3\frac{\pi p}{m}} - \&c.\right\},$$

Hence $(n=3)$ $\frac{1}{1^3}+\frac{1}{4^3}+\frac{1}{7^3}$, &c. $=\frac{13S_3}{27}+\frac{2\pi^3}{81\sqrt{(3)}}$,

$$\frac{1}{2^3}+\frac{1}{5^3}, \text{ \&c.} =\frac{13S_3}{27}-\frac{2\pi^3}{81\sqrt{(3)}};$$

$$(n=5)\ \frac{1}{1^5}+\frac{1}{4^5}+\frac{1}{7^5}, \text{ \&c.} =\frac{121}{243}S_5+\frac{2\pi^5}{729\sqrt{(3)}},$$

$$\frac{1}{2^5}+\frac{1}{5^5}, \text{ \&c.} =\frac{121S_5}{243}-\frac{2\pi^5}{729\sqrt{(3)}}.$$

Ex. 2. In (2), let $n=2$), then

$$\Sigma_{+\infty}^{-\infty}\frac{1}{(mx+p)^2}=\left(\frac{2\pi}{m}\right)^2\left\{\frac{1}{2}+\frac{1}{4}\frac{\cos\frac{2\pi p}{m}}{\sin^2\frac{\pi p}{m}}\right\}.$$

Hence $(m=6)$, $p=1)$,

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}, \text{ \&c.} =\frac{\pi^2}{9},$$

$(m=12,\ p=1)$,

$$\frac{1}{1^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{23^2}+\frac{1}{25^2}, \text{ \&c.} =\frac{\pi^2}{18}+\frac{\pi^2\sqrt{(3)}}{36}.$$

XVI. The following general infinite series, viz.

$$\frac{1}{2}\cdot\frac{1}{m^n}+\frac{1.3}{2.4}\cdot\frac{1}{(m+1)^n}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{(m+2)^n}+\text{\&c.},$$

can be summed in finite terms for successive integral and positive values of m and n.

(1) $(m=1,\ n=1)$,

$$\frac{1}{2}\cdot\frac{1}{1}+\frac{1.3}{2.4}\cdot\frac{1}{2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3}, \text{ \&c.} =2\log 2.$$

(2) $(m=1,\ n=2)$,

$$\frac{1}{2}\cdot\frac{1}{1^2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3^2}, \text{ \&c.} =S_2-2(\log 2)^2.$$

(3) $(m=1,\ n=3)$,

$$\frac{1}{2}\cdot\frac{1}{1^3}+\frac{1.3}{2.4}\cdot\frac{1}{2^3}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3^3}+\text{\&c.} =2S_3-2S_2\log 2+\frac{4}{3}(\log 2)^3.$$

&c. &c. &c.

(4) $(m=2,\ n=2)$,

$$\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1.3}{2.4}\cdot\frac{1}{3^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{4^2},\ \&c.=3-4\log 2.$$

(5) $(m=2,\ n=3)$,

$$\frac{1}{2}\cdot\frac{1}{2^3}+\frac{1.3}{2.4}\cdot\frac{1}{3^3}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{4^3},\ \&c.=7-8\log 2+4(\log 2)^2-2S_3,$$

&c. &c. &c.

(6) $(m=3,\ n=3)$,

$$\frac{1}{2}\cdot\frac{1}{3^3}+\frac{1.3}{2.4}\cdot\frac{1}{4^3}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{5^3},\&c.=\frac{869}{216}-\frac{40}{9}\log 2+\frac{8}{3}(\log 2)^2-\frac{4}{3}S_3,$$

&c. &c. &c.

The remaining theorems at present exhibited are instances of the evaluation of transcendents, and to these results, in conjunction with some of the preceding ones, I think I may confidently appeal, as exemplifying the great and peculiar power of my method and notation.

XVII. Let B be the representative of the Bernoulli numbers, and, as before, let

$$S_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n},\ \&c.\ ad\ inf.,$$

and $$D_n=S_n-1=\frac{1}{2^n}+\frac{1}{3^n},\ \&c.,$$

then $$1^{1^n}.2^{2^n}.3^{3^n}\ldots x^{x^n}\ (x\text{ infinite})=C_n x^{Fx}\varepsilon^{\phi x},$$

where, 1st, $\log C_n$ can always be expressed in finite terms when n is even, and by means of a converging series when n is odd.

2nd. $$Fx=\frac{x^{n+1}}{n+1}+\frac{x^n}{2}+\frac{n}{2}B_2x^{n-1}$$
$$+\frac{n(n-1).(n-2)}{2.3.4}B_4x^{n-3}+\ldots+B_{n+1},$$

the last term B_{n+1} only holding good when $n>0$; if $n=0$, the last term is $\frac{1}{2}$.

3rd. $$\phi x = -\frac{1}{n+1}\left\{\frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n}{2}B_2 x^{n-1} + \frac{n(n-1).(n-2)}{2.3.4}x^{n-3} + \ldots + B_n x\right\}$$

$$+\frac{1}{n+1}\left[\frac{x^n}{2} + \left(\frac{n+1}{1} - \frac{1}{2}\right)B_2 x^{n-1} + \left\{\frac{(n+1).n(n-1)}{1.2.3} - \frac{1}{2}\cdot\frac{(n+1).n}{1.2} + \frac{1}{3}\cdot\frac{n+1}{1} - \frac{1}{4}\right\}B_4 x^{n-3} + \&c.\right],$$

the law of formation of terms being evident, observing however that ϕx contains no term which does not involve x.

Ex. 1. $(n=0)$, $\log C_0 = \frac{1}{2}\log 2\pi$, $Fx = x + \frac{1}{2}$, $\phi x = -x$, hence $$1.2.3\ldots x\ (x \text{ infinite}) = \sqrt{(2\pi)}\, x^{x+\frac{1}{2}}\varepsilon^{-x}.$$

Ex. 2. $(n=1)$,

$$\log C_1 = \frac{13}{24} - \frac{1}{4}\log 2 - \frac{1}{2}\left(\frac{\gamma}{3} + \frac{D_3}{5} + \frac{D_5}{7}, \&c.\right) = \cdot 248754,^*$$

$$Fx = \frac{x^2}{2} + \frac{x}{2} + \frac{1}{12},\quad \phi x = -\frac{x^2}{4},$$

hence $$1^1.2^2.3^3\ldots x^x\ (x \text{ inf.}) = \varepsilon^{\cdot 248754 - \frac{x^2}{4}}.x^{\frac{x^2}{2} + \frac{x}{2} + \frac{1}{12}}.$$

Ex. 3. $(n=2)$, $\log C_2 = \dfrac{S_3}{4\pi^2}$, and

$$1^{1^2}.2^{2^2}.3^{3^2}\ldots x^{x^2}\ (x \text{ inf.}) = \varepsilon^{-\frac{x^3}{9} + \frac{x}{12} + \frac{S_3}{4\pi^2}}.x^{\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}}.$$

Ex. 4. $(n=4)$,

$$1^{1^4}.2^{2^4}.3^{3^4}\ldots x^{x^4}\ (x \text{ inf.}) = \varepsilon^{-\frac{x^5}{25} + \frac{x^3}{12} - \frac{13x}{360} - \frac{3S_5}{4\pi^4}}.x^{\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}}.$$

XVIII. An indefinite number of relations may be established among the D numbers, of which the following appear to have a special importance.

* It is satisfactory to me to find that so able an analyst as Mr. Jeffery has calculated this constant by other methods, since not only is the accuracy of the above formula thus tested, but the correctness (about which, it is true, I had no doubt) of my method and its results is also so far independently established. See No. 18, pp. 97, 98.

Let $$a_n = \frac{D_2}{n+1} + \frac{D_4}{n+3} + \frac{D_6}{n+5}, \text{ \&c.},$$

then the values of a_1, a_2, a_3, &c. can be successively determined.

$$(1)\ a_1\left(= \frac{D_2}{2} + \frac{D_4}{4} + \frac{D_6}{6}, \text{ \&c.}\right) = \frac{1}{2}\log 2,$$

(a well-known formula).

$$(2)\ a_2\left(= \frac{D_2}{3} + \frac{D_4}{5} + \frac{D_6}{7}, \text{ \&c.}\right) = \frac{3}{2} - \log 2 - \frac{1}{2}\log\pi.$$

$$(3)\ a_3\left(= \frac{D_2}{4} + \frac{D_4}{6} + \frac{D_6}{8}, \text{ \&c.}\right) = \frac{3}{4} - \frac{1}{2}\log\pi.$$

$$(4)\ a_4\left(= \frac{D_2}{5} + \frac{D_4}{7} + \frac{D_6}{9}, \text{ \&c.}\right) = \frac{3}{2} - \log 2 - \frac{1}{2}\log\pi - \frac{3S_3}{4\pi^2}.$$

$$(5)\ a_5\left(= \frac{D_2}{6} + \frac{D_4}{8} + \frac{D_6}{10}, \text{ \&c.}\right) = \frac{7}{8} - \frac{1}{2}\log\pi - \frac{3S_3}{2\pi^2}.$$

$$(6)\ a_6\left(= \frac{D_2}{7} + \frac{D_4}{9} + \frac{D_6}{11}, \text{ \&c.}\right) = \frac{49}{30}$$

$$- \log 2 - \frac{1}{2}\log\pi - \frac{5S_3}{2\pi^2} + \frac{15S_5}{4\pi^4}.$$

$$(7)\ a_7\left(= \frac{D_2}{8} + \frac{D_4}{10} + \frac{D_6}{12}, \text{ \&c.}\right) = 1 - \frac{1}{2}\log\pi - \frac{15S_3}{4\pi^2} + \frac{45S_5}{4\pi^4}.$$

$$(8)\ a_8\left(= \frac{D_2}{9} + \frac{D_4}{11} + \frac{D_6}{13}, \text{ \&c.}\right) = \frac{367}{210} - \log 2 - \frac{1}{2}\log\pi$$

$$- \frac{21S_3}{4\pi^2} + \frac{105S_5}{4\pi^4} - \frac{315S_7}{8\pi^6}.$$

&c. &c. &c.

Similarly, let $$b_n = \frac{\gamma}{n} + \frac{D_3}{n+2} + \frac{D_5}{n+4} + \text{\&c.},$$

then $$b_1\left(= \frac{\gamma}{1} + \frac{D_3}{3} + \frac{D_5}{5}, \text{ \&c.}\right) = 1 - \frac{1}{2}\log 2,$$

$$b_2\left(= \frac{\gamma}{2} + \frac{D_3}{4} + \frac{D_5}{6}, \text{ \&c.}\right) = \frac{1}{2}\log 2,$$

$$3b_2 - 2b_4 \left\{ = 3\left(\frac{\gamma}{3} + \frac{D_3}{5} + \frac{D_5}{7}, \text{\&c.}\right) - 2\left(\frac{\gamma}{4} + \frac{D_3}{6} + \frac{D_5}{8}, \text{\&c.}\right)\right\} = \frac{25}{12} - \frac{5}{2}\log 2.$$

In the same manner, relations between each pair of quantities b_5, b_6; b_7, b_8; &c., are determined, but the quantities b_2, b_4, b_5, &c. cannot be separately obtained.

The importance of these a and b numbers appears from the fact that the values of $\log C_n$ in the preceding theorem depend upon them.

XIX. A large number of non-converging series can be evaluated, which possess the property, common to such series, of approximating to the true value as long as the terms converge.

Thus $\dfrac{B_{2n}}{1.2\ldots 2n} + \dfrac{B_{2n+2}}{3.4\ldots\{(2n+2)\}} + \dfrac{B_{2n+4}}{5.6\ldots(2n+4)}$, &c.

can be evaluated.

Ex. 1. $(n=1)$,

$$\frac{B_2}{1.2} + \frac{B_4}{3.4} + \frac{B_6}{5.6} + \text{\&c.} = 1 - \frac{1}{2}\log 2\pi = \frac{1}{2} + (B+1)\log(B+1).$$

Similarly, $\dfrac{A_{2n+1}}{1.2\ldots(2n+1)} + \dfrac{A_{2n+3}}{3.4\ldots(2n+3)} + \dfrac{A_{2n+5}}{5.6\ldots(2n+5)}$, &c.

can also be evaluated, where $A_{2n+1} = -\dfrac{2^{2n+2}-1}{n+1}B_{2n+2}$.

Ex. 2 $(n=0)$,

$$\frac{A_1}{1} + \frac{A_3}{3} + \frac{A_5}{5}, \text{\&c.} \{= \log(1+A)\} = \log\frac{2}{\pi}.$$

Ex. 3. $\log(1+B) = -\gamma$, where $\gamma = {\cdot}5772157$,*

hence $$\frac{B_2}{2} + \frac{B_4}{4} + \frac{B_6}{6}, \text{\&c.} = \gamma - \frac{1}{2}.$$

Ex. 4.

$$\log\Gamma(1+B)\left(= -\gamma B_1 + \frac{S_2}{2}B_2 + \frac{S_4}{4}B_4, \text{\&c.}\right) = \frac{1}{2}\log(2\pi) - \frac{1}{2}.$$

Ex. 5.

$$\log\Gamma(1+A)\left(= -\gamma A_1 - \frac{S_3}{3}A_3 - \frac{S_5}{5}A_5 - \text{\&c.}\right) = \frac{1}{2}\log\left(\frac{\pi}{2}\right),$$

&c. &c. &c.

* See De Morgan's *Differential and Integral Calculus*, p. 578.

XX. Lastly, the gamma function with all its properties is capable of a very elegant (as it seems to me) and extensive generalization, as an instance of which, Legendre's Theorem, viz.,

$$\Gamma x . \Gamma\left(x+\frac{1}{n}\right) . \Gamma\left(x+\frac{2}{n}\right) \ldots \Gamma\left(x+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}n-1} . n^{\frac{1}{2}-nx} . \Gamma nx$$

may be generalized as follows:

Let $\qquad \Gamma_m (1+x) = 1^{1^m} . 2^{2^m} . x^{x^m}$;

therefore $\qquad (m=0)\, \Gamma_0 (1+x) = 1.2 \ldots x = \Gamma(1+x)$,

so that when $m=0$ the zero sub-index may be withdrawn. Then

$$\left\{\Gamma_m (x) . \Gamma_m\left(x+\frac{1}{n}\right) . \Gamma_m\left(x+\frac{2}{n}\right) \ldots \Gamma_m\left(x+\frac{n-1}{n}\right)\right\}^{n^m}$$

$$= H_m n^{-\psi(nx)} . \Gamma_m (nx),$$

where, 1st. $\log H_m$ depends for its value on $\log C_m$ of Theorem XVII., and can be expressed in finite terms when m is even, and by means of a converging series when m is odd.

2nd. $$\psi(nx) = \frac{1}{m+1}\left\{(nx)^{m+1} + \frac{1}{m+1} B_1 (nx)^m\right.$$

$$\left. + \frac{(m+1).m}{1.2} B_2 (nx)^{m-1} + \frac{(m+1).m.(m-1).(m-2)}{1.2.3.4} B_4 (nx)^{m-3} + \&c.\right\}.$$

Let $m=0$, then $H_0 = C_0^{\,n-1} = (2\pi)^{\frac{1}{2}(n-1)}$, $\psi(nx) = nx - \frac{1}{2}$, and we have Legendre's Theorem,

$$\Gamma x . \Gamma\left(x+\frac{1}{n}\right) \ldots \Gamma\left(x+\frac{n-1}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)} . n^{\frac{1}{2}-nx} . \Gamma nx.$$

In the above generalized theorem, for x put $x+\frac{1}{n}$, then

$$\left\{\Gamma_m\left(x+\frac{1}{n}\right) . \Gamma_m\left(x+\frac{2}{n}\right) \ldots \Gamma_m (x+1)\right\}^{n^m} = H_m n^{-\psi(nx+1)} . \Gamma_m (nx+1);$$

therefore dividing, we have

$$\left\{\frac{\Gamma_m (x+1)}{\Gamma_m (x)}\right\}^{n^m} = n^{-\psi(nx+1)+\psi(nx)} . \frac{\Gamma_m (nx+1)}{\Gamma_m (nx)},$$

i.e., $\qquad (x^{x^m})^{n^m} \left(= x^{nx^m}\right) = n^{-\psi(nx+1)+\psi(nx)} . nx^{(nx)^m}.$

Now $\psi(nx)$ (by representative notation) $= \dfrac{1}{m+1}(nx+B)^{m+1}$,

and $$\psi(nx+1) = \frac{1}{m+1}(nx+B+1)^{m+1};$$

therefore $$\psi(nx+1) - \psi(nx) = (nx)^m;$$

therefore $$x^{(nx)^m} = n^{-(nx)^m}.(nx)^{(nx)^m},$$

which is the case, and thus a test has been applied which shews the correctness of the theorem.

It is evident that by giving different positive integral values to n in the above generalization of Legendre's Theorem, the transcendents

$$\Gamma_m\left(\frac{1}{2}\right),\ \Gamma_m\left(\frac{1}{3}\right).\Gamma_m\left(\frac{2}{3}\right),\ \ldots\Gamma_m\left(\frac{1}{n}\right).\Gamma_m\left(\frac{2}{n}\right)\ldots\Gamma_m\left(1-\frac{1}{n}\right)$$

can be evaluated, and it is thus highly probable that *all* the properties of the ordinary gamma function can be analogously exhibited in a generalized form.

NOTE. The demonstration of the last four theorems and classes of formulæ will appear in the remaining part of the present chapter.

5. The preceding list of theorems and formulæ, most of which I suppose to be perfectly new, has been given for the purpose of exhibiting, in brief compass, a body of results which may serve to recommend the notation through which they have been obtained,—a notation which, I confidently believe must, in course of time, from its perfect simplicity and unrivalled power, come into general use. It may certainly, I think, be regarded as tending to give to analysis the eminently desirable qualities of unity, compactness, and simplicity; and since it is of wholly unlimited application, it is perhaps in the perfecting of this notation, as a proper tool to work with, that a large extension of modern analysis may not unreasonably be expected. If one like myself can obtain by its use, in whatever direction it may be applied, novel results of great generality and symmetry, what may it not prove in the hands of the able and accomplished analysts who adorn our age and country?

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ON OSTROGRADSKY'S HYDROSTATICAL SHELL.

By William Walton, M.A., Trinity College.

From a Memoir by Ostrogradsky, entitled *Sur un cas singulier de l'équilibre des fluides incompressibles*, published, in the year 1838, in the *Mémoires de l'Académie Impériale des Sciences de Saint-Pétersbourg*, I extract the following passage, in which a condition of fluid equilibrium is described which tends to shew that the ordinary theory of hydrostatics is not universally true.

"Supposons que le liquide forme une couche sphérique d'une épaisseur quelconque et dont chaque molécule soit attirée vers le centre par une force proportionnée à une fonction de la distance de la molécule au centre: l'équilibre aura nécessairement lieu. Car les molécules, situées à une même distance du centre d'attraction, ne peuvent se mouvoir que toutes de la même manière; si l'une d'elles s'approche du centre, toutes les autres doivent s'en approcher, et de la même distance, et elles ne peuvent pas s'en approcher de manière que toutes celles, situées sur une même surface sphérique décrite du centre d'attraction, conservent le même mouvement; car il en résulterait une diminution du volume liquide. Ainsi le liquide restera en équilibre: mais il est évident que la force, qui attire chaque molécule située à la surface intérieure de la couche, est dirigée en dehors de la masse liquide. Appelons $f(r)$ l'attraction, a le rayon de la surface inférieure, b le rayon de la surface supérieure: nous aurons

$$dp = -f(r)\,dr:$$

donc
$$p = -\int_b^r f(r)\,dr = \int_r^b f(r)\,dr:$$

donc la pression sur la surface inférieure sera $\int_a^b f(r)\,dr$, et cette pression est certainement différente de zéro, ce qui est encore contraire à ce qu'on avait généralement admis.

Voici donc un cas singulier de l'équilibre qui échappe à la théorie connue des liquides, et qui autorise à penser que cette théorie n'a pas une étendue convenable."

I propose to offer a few remarks on this singular case of fluid equilibrium, not only because it is in itself curious as a speculative question, but also because a right interpretation of such equilibrium, although, as being unstable, only theoretically possible, may tend to awaken criticism on the fundamental principles of practical hydrostatics.

Instead of taking into consideration at once a thick fluid shell, as Ostrogradsky has done, let us commence with the consideration of an indefinitely thin one. Let t denote the tangential pressure of this thin spherical shell. Then, by reasoning substantially coincident with that which presents itself in the investigation of the tension of flexible surfaces exposed to the action of fluid, we shall see that

$$t = \tfrac{1}{2} r f(r).$$

Moreover, the pressure on the outer surface of the shell is zero, while that on the inner cannot exceed the infinitesimal quantity $f(r)\,dr$, where dr is the thickness of the shell.

Next conceive a number of such thin shells to be aggregated around the same centre of force so as to form a thick spherical shell, like the one considered by Ostrogadsky, the aggregation being so effected that, as is evidently possible, the contact of the successive infinitesimal shells may take place without mutual pressure. The method of structure of Ostrogradsky's shell, which I have suggested, and which is manifestly possible in a theoretical sense, involves results not in accordance with those obtained by Ostrogradsky as consequences of the ordinary equations of hydrostatics: while, by the application of these equations, he obtains, for the pressure at the *surface inférieure* of his shell, the expression

$$\int_a^b f(r)\,dr,$$

I have shewn that it need not exceed an infinitesimal quantity.

It may easily be seen also that the normal pressure at the lower surface of the shell may be finite, while the tangential pressure is infinite. Suppose in fact the infinitesimal shell of longest diameter to be so related to the next in magnitude that their relative contact may involve mutual pressure of such magnitude as to destroy the tangential pressure of the larger shell: the normal pressure will then be transmitted to the second shell; and the same arrangement may be perpetuated till we arrive at the shell of least diameter. Under this arrangement the normal pressure at the *surface inférieure* of Ostrogradsky's shell will be in

accordance with his result; while the tangential pressure will be infinite.

The aggregate shell may be constructed in an infinite number of other ways, mechanically different in regard to internal pressures, although geometrically the same. The problem of internal pressures in Ostrogradsky's shell is in fact indeterminate.

It is now time to consider why results such as these do not follow from the ordinary hydrostatical equations, and are in fact at variance with the fundamental principle of hydrostatics. I proceed to shew why these ordinary equations are not applicable to a case such as Ostrogradsky's shell. The fundamental principle of hydrostatics, viz., *the equality of fluid pressure at any point of a fluid in all directions*, is either proved theoretically or confirmed experimentally. Since experiment is inapplicable in a case of unstable equilibrium, we may confine our attention to theoretical demonstrations. In all these demonstrations two assumptions are made, (1) that the fluid pressure at any point is normal to any plane element of the fluid at that point, and (2) that the pressure on the element varies as the area of the element. Now, in the aggregation of the infinitesimal shells, as I have conceived them to be put together, neither of these assumptions is necessarily true, the former because to an infinitesimal fluid area, taken at random, the pressure (as is evident from my mode of aggregation) is generally oblique, the latter because the pressure is generally discontinuous in passing from shell to shell.

We may therefore conclude that the theoretic demonstration of the equality of fluid pressure in all directions at any point is unsound in regard to unstable equilibrium. Why it should be sound in relation to stable equilibrium is not, I think, *à priori*, obvious. The confirmation of the truth of results, obtained on the principle of the equality of fluid pressure, by an appeal to experiment, no doubt confirms the truth of the two assumptions on which the demonstration of the principle is founded. I do not think however that these assumptions ought to be regarded, as they evidently are by writers of hydrostatical treatises, as evident truths, but rather as hypotheses not perhaps more evident than the proposition to the demonstration of which they are applied. At any rate, if they be more evident, it would be necessary to give a logical reason why, not necessarily true in regard to unstable equilibrium, they should be evidently true always in regard to stable.

By a continued aggregation of the infinitesimal fluid shells of decreasing diameters around the centre of force, we shall eventually construct a spherical volume of fluid, endowed with all the mechanical properties of Ostrogradsky's thick shell. Thus we have a complete sphere of fluid in equilibrium in which "the principle of the equality of fluid pressure at any point in all directions" is not true. Shall we call the equilibrium stable or unstable? The proper answer I conceive to be, that the fluid is stable in regard to the geometry of the fluid, and unstable in regard to the relative mechanical action of its elements. A slight jar to the system would, I conceive, without disturbing the geometry of the fluid, transform the internal forces from a system of instability to one of stability, and place the internal pressure at each point in its orthodox state.

July 2, 1861.

DETERMINATION OF THE FORMS OF THE ROOTS OF SOLVIBLE QUINTIC EQUATIONS WHOSE COEFFICIENTS ARE FUNCTIONS OF A VARIABLE.

By the Rev. GEORGE PAXTON YOUNG, M.A., Professor of Logic and Metaphysics in Knox College, Toronto, Canada, West.

IN a paper on the "Exact Resolution of Algebraical Equations of every Degree in all the Solvible Cases," which appeared in a recent number of this *Journal*,* it is proved that a solvible irreducible equation of the m^{th} degree, which wants the second term, and has its coefficients rational, m being a prime number, has all its roots contained in the expression

$$x_1 = A_1 Y^{\frac{1}{m}} + A_2 Y^{\frac{2}{m}} + \ldots + A_{m-1} Y^{\frac{m-1}{m}} \ldots\ldots\ldots (1),$$

where A_1, A_2, &c. are expressions involving only such surds as are subordinates of the surd $Y^{\frac{1}{m}}$; the quantities

$$A_1^m Y,\ A_2^m Y^2,\ \ldots,\ A_{m-1}^m Y^{m-1} \ldots\ldots\ldots\ldots (2)$$

* See Vol. IV., p. 341,—ED.

being the roots of an equation of the $(m-1)^{\text{th}}$ degree, whose coefficients are rational. Take then the solvible irreducible quintic

$$x^5 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0 \;\ldots\ldots\ldots\ldots (3),$$

whose coefficients are rational functions of a variable p. In this case, equation (1) becomes

$$x_1 = A_1 Y^{\frac{1}{5}} + \&c.;$$

or, since the factor A_1, in the first term, may be thrown under the main radical, and may thus be considered equal to unity,

$$x_1 = Y^{\frac{1}{5}} + A_2Y^{\frac{2}{5}} + A_3Y^{\frac{3}{5}} + A_4Y^{\frac{4}{5}} \;\ldots\ldots\ldots (4).$$

The expressions (2) become

$$Y,\ A_2{}^5Y^2,\ A_3{}^5Y^3,\ A_4{}^5Y^4 \ldots\ldots\ldots\ldots\ldots (5).$$

What follows will best be arranged in Propositions, which (to prevent confusion) may be numbered in continuation of those in the paper above referred to, so that the first Proposition here will be Prop. IV.

Proposition IV.

The quintic (3) has its roots of one or other of the following classes:

Class First, $\quad x_1 = Y^{\frac{1}{5}} + A_2Y^{\frac{2}{5}} + A_3Y^{\frac{3}{5}} + A_4Y^{\frac{4}{5}} \ldots\ldots\ldots (6);$

Class Second, $x_1 = 2\sqrt{(b)}\cos\dfrac{\theta}{5} + 2b\sqrt{(\beta)}\cos\left(\dfrac{2\theta}{5} + \phi\right) \ldots (7);$

Class Third, $\quad x_1 = 2\sqrt{\{e + f\sqrt{(U)}\}}\cos\dfrac{\theta}{5} + 2\sqrt{\{e - f\sqrt{(U)}\}}\cos\dfrac{\phi}{5}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$

We announce these forms, not for the purpose of establishing them in the present proposition, but as the results finally to be arrived at; and we will now merely explain the quantities which appear in the values of the roots. In *Class First*, the terms Y, A_2, A_3, A_4 are rational. In *Class Second*, the quantities

$$b,\ \beta,\ \frac{\tan\theta}{\tan\phi},\ \sqrt{(b)}\cos\theta,\ \sqrt{(\beta)}\cos\phi \ldots\ldots\ldots (9)$$

are rational. In *Class Third*, e, f, U, together with the expressions

$$\left.\begin{array}{c} b^5 \cos\theta + \beta^5 \cos\phi, \\ b^2\beta \cos\frac{2\theta+\phi}{5} + \beta^2 b \cos\frac{\theta-2\phi}{5}, \\ b^3\beta \cos\frac{3\theta-\phi}{5} + \beta^3 b \cos\frac{\theta+3\phi}{5}, \\ (3\beta^3 b^2 + 2\beta^4 b)\cos\frac{\theta-2\phi}{5} + (3b^2\beta^3 + 2b^4\beta)\cos\frac{2\theta+\phi}{5} \end{array}\right\} \ldots\ldots(10)$$

are rational: b^5 being put for $e+f\surd(U)$, and β^5 for $e-f\surd(U)$. It will appear, by actual trial, that, whatever be the value (subject to the conditions mentioned) of the unknown quantities on the right-hand side of equations (6), (7), and (8), these equations exhibit the roots of quintics with rational coefficients; the five roots being found, in Class First, by taking the five values of the surd $Y^{\frac{1}{5}}$, and in Classes Second and Third, by writing

$$\frac{\theta}{5}, \frac{\theta+2\pi}{5}, \frac{\theta+4\pi}{5}, \frac{\theta+6\pi}{5}, \frac{\theta+8\pi}{5} \ldots\ldots (11)$$

successively instead of $\frac{\theta}{5}$. It will be proved that the quintic (3) does not admit of algebraical solution, unless it belongs to one of the Classes specified; and it will also be seen, as we proceed, how the roots of the Second and Third Classes can be expressed in algebraical functions. As the reader may wish to see definite instances of these two last kinds of roots, we give the following numerical examples:

Class Second, $x_1 = \{2+\surd(3)\}^{\frac{1}{5}} + \{2+\surd(3)\}^{\frac{2}{5}}$
$+ \{2-\surd(3)\}\,[\{2+\surd(3)\}^{\frac{3}{5}} + \{2+\surd(3)\}^{\frac{4}{5}}] \ldots (12)$;

Class Third, $x_1 = Y^{\frac{1}{5}} - [8+6\surd(2) + \{3+2\surd(2)\}\surd\{10-\surd(2)\}]\,Y^{\frac{2}{5}}$
$+ \{20+14\surd(2)\}\,[8+6\surd(2) - \{3+2\surd(2)\}\surd\{10-\surd(2)\}]\,Y^{\frac{3}{5}}Y_2$
$+ 4\{17+12\surd(2)\}\,Y_2 Y^{\frac{4}{5}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$,

where Y and Y_2 have the values

$$Y = 3 + \surd(2) - \frac{4+27\surd(2)}{28}\surd\{10-\surd(2)\},$$

$$Y_2 = 3 + \surd(2) + \frac{4+27\surd(2)}{28}\surd\{10-\surd(2)\}.$$

Before proceeding to the next Proposition, we would observe that an irreducible biquadratic has its roots of two forms:

First Form, $y_1 = B + \sqrt{(C)} + \sqrt{(D)} + E\sqrt{(C)}\sqrt{(D)}$......(14);

Second Form, $y_1 = E + \sqrt{(Q)}$..................................(15).

In the first form, B is rational; and $\sqrt{(C)}$ and $\sqrt{(D)}$ are surds neither of which is subordinate (Def. 3)* to the other; C and D not being necessarily rational; and E involving only surds subordinate to $\sqrt{(C)}$ and $\sqrt{(D)}$. In the second form, $\sqrt{(Q)}$ is the only surd in the root which is not subordinate to any other; E involving only surds which are subordinates of $\sqrt{(Q)}$. In (14) and (15), we understand the root y_1 to be in a simple (Def. 7) form, and to have no two surds similarly (Def. 6) involved in it.

Proposition V.

The biquadratic, whose roots are the expressions forming the series (5), has *not* its roots of the form (14).

Suppose, if possible, that the biquadratic has a root of the form (14). The four roots of the biquadratic are found (compare Cor. 2, Def. 10) by taking the two values of $\sqrt{(C)}$ in connection with the two values of $\sqrt{(D)}$. But one of these roots [see (5)] is Y. We may therefore put

$$\left.\begin{array}{ll} & Y = B + \sqrt{(C)} + \sqrt{(D)} + E\sqrt{(C)}\sqrt{(D)}, \\ \text{Put also} & Y_1 = B + \sqrt{(C)} - \sqrt{(D)} - E\sqrt{(C)}\sqrt{(D)}, \\ \text{and} & Y_2 = B - \sqrt{(C)} + \sqrt{(D)} - E\sqrt{(C)}\sqrt{(D)} \end{array}\right\}\text{.....(16).}$$

It is plain that the terms

$$Y_1,\ a_2^{\,5}Y_1^{\,2},\ a_3^{\,5}Y_1^{\,3},\ a_4^{\,5}Y_1^{\,4}\text{...............(17)}$$

are [a_2, a_3, a_4 being what A_2, A_3, A_4 in (5) become, when we pass from Y to Y_1 by changing the sign of $\sqrt{(D)}$] the roots of the same biquadratic which has the terms in (5) for its roots; and therefore the terms in (17) are equal to those in (5), in a certain order, each to each. Suppose, if possible, that $Y_1 = Y$. Then, from (16),

$$2\sqrt{(D)} + 2E\sqrt{(C)}\sqrt{(D)} = 0\text{..........(18),}$$

* Our references to Definitions and to Propositions I., II., III. are to the paper of this *Journal* above referred to.

which is impossible, since (Corollaries 2 and 3, Def. 7) such an equation would require the coefficients of $\surd(D)$ and $\surd(C)\,\surd(D)$ to vanish separately. Therefore Y is not equal to the first term in (17). Suppose next that

$$Y = a_2{}^5 Y_1{}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

Then, if $a_2 = h + k\surd(D)$, where h and k are clear of the surd $\surd(D)$, we have

$$B + \surd(C) + \surd(D) + E\surd(C)\,\surd(D)$$
$$= \{h + k\surd(D)\}^5 \{B + \surd(C) - \surd(D) - E\surd(C)\,\surd(D)\}^2 \ldots (20).$$

When this equation is arranged according to the roots $\surd(C)$, $\surd(D)$, $\surd(C)\,\surd(D)$, the coefficients of these roots must, as in (18), vanish separately. Hence equation (20) implies that

$$B + \surd(C) - \surd(D) - E\surd(C)\,\surd(D)$$
$$= h - k\surd(D)\}^5 \{B + \surd(C) + \surd(D) + E\surd(C)\,\surd(D)\}^2;$$

therefore $$Y_1 = A_2{}^5 Y^2.$$

Hence, from (19), we have

$$Y^{\frac{1}{5}} = w a_2 A_2{}^2 Y^{\frac{4}{5}},$$

where w is a fifth root of unity. But this is impossible, since (Corollaries 2 and 3, Def. 7) such an equation would require the coefficients of $Y^{\frac{1}{5}}$ and $Y^{\frac{4}{5}}$ to vanish separately. Therefore Y is not equal to the second term in (17). In the same manner it can be proved that Y is not equal to the third term in (17). Therefore it must be equal to the last term in (17). But in the same way it can be proved that Y_2 is equal to the last term in (17). Therefore $Y = Y_2$: which involves the same impossibility already proved to attach to the hypothesis that $Y = Y_1$. Hence it cannot be true that the biquadratic whose roots form the series (5) has its roots of the form (14).

Proposition VI.

If the biquadratic, whose roots form the series (5), can be broken into two equations, the one a cubic with rational coefficients, and the other a rational simple equation, the quintic (3) belongs to what, in Prop. IV., we termed *Class First.*

For, by hypothesis, one of the terms in (5), which we may take as the first, is rational. That is, Y is rational; and $Y^{\frac{1}{5}}$ is a surd of the first (Def. 2) order. But A_2, A_3, A_4, in (4) and (5), involve no surds which are not subordinates of $Y^{\frac{1}{5}}$. Hence, since $Y^{\frac{1}{5}}$ has no subordinates; A_2, A_3, A_4 are rational; so that, in (4), all the quantities, Y, A_2, A_3, A_4, being rational, the quintic (3) belongs to what we called *Class First*. It is hardly necessary to remark, that, whatever be the values of Y, A_2, A_3, A_4, the expression x_1 in (4) is the root of a quintic with rational coefficients. This is, in fact, a particular case of the wide general truth established in Cor. 1, Def. 8.

Proposition VII.

If the biquadratic whose roots form the series (5) have no rational factor of the first degree, but can be broken into two rational quadratic factors, the quintic (3) belongs to what we have termed *Class Second*.

For, in this case, each of the terms in (5) is the root of an irreducible quadratic, so that we may put

$$Y = B + \sqrt{(C)} \qquad (21),$$

where B and C are rational. Also, if

$$Y_1 = B - \sqrt{(C)} \qquad (22);$$

and if a_2, a_3, a_4 be what A_2, A_3, A_4 become when we pass from Y to Y_1 by changing the sign of $\sqrt{(C)}$; the terms

$$Y_1,\ a_2^5 Y_1^2,\ a_3^5 Y_1^3,\ a_4^5 Y_1^4 \qquad (23)$$

are (as in Prop. v.) equal to those forming the series (5), each to each, in a certain order. Still further, it can be proved, exactly as in Prop. v., that Y is not equal to any of the first three terms in (23); so that

$$\left.\begin{array}{l} Y = a_4^5 Y_1^4;\ \text{therefore } YY_1 = a_4^5 Y_1^5. \\ \text{In like manner, } Y_1 = A_4^5 Y^4;\ \text{therefore } YY_1 = A_4^5 Y^5 \end{array}\right\} \ldots (24).$$

Therefore, if $A_4 = D + E\sqrt{(C)}$, D and E being rational, we have

$$\{D + E\sqrt{(C)}\}^5 \{B + \sqrt{(C)}\}^5 = \{D - E\sqrt{(C)}\}^5 \{B - \sqrt{(C)}\}^5:$$

an equation which implies that

$$\left.\begin{array}{rl} & A_4 = D + E\sqrt{(C)} = P\{B - \sqrt{(C)}\} = PY_1 \\ \text{and} & a_4 = D - E\sqrt{(C)} = P\{B + \sqrt{(C)}\} = PY \end{array}\right\} \ldots (25),$$

where P is clear of the surd $\sqrt{(C)}$, and is therefore (Def. 1) rational. By comparing (25) and (24), we get

$$(YY_1)^{\frac{1}{5}} = PYY_1 = P(B^2 - C).$$

So that $(YY_1)^{\frac{4}{5}}$, or $(B^2 - C)^{\frac{4}{5}}$ is rational. Put

$$(B^2 - C)^{\frac{1}{5}} = b.$$

$$\left.\begin{array}{ll}\text{Then} & Y = B + \sqrt{(B^2 - b^5)} = \sqrt{(b^5)}\{\cos\theta + \sqrt{(-1)}\sin\theta\} \\ \text{and} & Y_1 = B - \sqrt{(B^2 - b^5)} = \sqrt{(b^5)}\{\cos\theta - \sqrt{(-1)}\sin\theta\}\end{array}\right\}\ldots(26).$$

Therefore $\quad Y^{\frac{1}{5}} + A_4 Y^{\frac{4}{5}} = Y^{\frac{1}{5}} + Y_1^{\frac{1}{5}} = 2\sqrt{(b)}\cos\dfrac{\theta}{5}.$

In like manner it can be proved that

$$A_2 Y^{\frac{2}{5}} + A_3 Y^{\frac{3}{5}} = A_2 Y^{\frac{2}{5}} + a_2 Y_1^{\frac{2}{5}} = 2b\sqrt{(\beta)}\cos\left(\frac{2\theta}{5} + \phi\right),$$

β being, like b, rational. Hence

$$x_1 = 2\sqrt{(b)}\cos\frac{\theta}{5} + 2b\sqrt{(\beta)}\cos\left\{\frac{2\theta}{5} + \phi\right\}\ldots\ldots(27).$$

This is the equation given in (7). Also, from (26), $\sqrt{(b)}\cos\theta = \dfrac{B}{b^2}$; so that $\sqrt{(b)}\cos\theta$ is rational. Again, if $A_2 = G + H\sqrt{(C)}$, we have $\sqrt{(\beta)}\cos\phi = G$; so that $\sqrt{(\beta)}\cos\phi$ is rational. Once more, $\dfrac{\tan\phi}{\tan\theta} = \dfrac{BH}{G}$; so that $\dfrac{\tan\phi}{\tan\theta}$ is rational. Thus all the expressions in (9) are rationial, and hence the quintic (3) belongs to Class Second.

Cor. It is easily seen that the form of the root x_1, in algebraical functions, is

$$x_1 = Y^{\frac{1}{5}} + \{G + H\sqrt{(C)}\}\,Y^{\frac{2}{5}}$$
$$+ \frac{\{G - H\sqrt{(C)}\}\{B - \sqrt{(C)}\}}{b^3}\,Y^{\frac{3}{5}} + \frac{\{B - \sqrt{(C)}\}}{b^4}\,Y^{\frac{4}{5}}\ldots(28),$$

where, as above, $C = B^2 - b^5$; and $Y = B + \sqrt{(C)}$. Any values whatsoever of the unknown quantities in (28) render x_1 the root of a quintic with rational coefficients, so that equation (27) or equation (28) is the solution in the most general case of quintics belonging to Class Second.

Proposition VIII.

If the biquadratic whose roots form the series (5) be irreducible, the quintic (3) belongs to what we termed *Class Third.*

Since (Prop. v.) the roots of the biquadratic are not of the form (14), they must be of the form (15). This form, more fully exhibited, is

$$y_1 = E + \sqrt{(Q)} = F + G\sqrt{(U)} + \sqrt{\{H + K\sqrt{(U)}\}};$$

where F, G, H, K, U, are rational. Put

$$\left.\begin{aligned} Y &= F + G\sqrt{(U)} + \sqrt{\{H + K\sqrt{(U)}\}} = M\{\cos\theta + \sqrt{(-1)}\sin\theta\}, \\ Y_1 &= F - G\sqrt{(U)} + \sqrt{\{H - K\sqrt{(U)}\}} = N\{\cos\phi + \sqrt{(-1)}\sin\phi\}, \\ Y_2 &= F + G\sqrt{(U)} - \sqrt{\{H + K\sqrt{(U)}\}} = M\{\cos\theta - \sqrt{(-1)}\sin\theta\}, \\ Y_3 &= F - G\sqrt{(U)} - \sqrt{\{H - K\sqrt{(U)}\}} = N\{\cos\phi - \sqrt{(-1)}\sin\phi\}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(29).$$

Now, if a_2, a_3, a_4, be what A_2, A_3, A_4, become when we pass from Y to Y_2, by changing the sign of $\sqrt{\{H + K\sqrt{(U)}\}}$, leaving the sign of $\sqrt{(U)}$ unchanged, the terms

$$Y_2,\ a_2{}^5 Y_2^2,\ a_3{}^5 Y_2^3,\ a_4{}^5 Y_2^4 \ldots\ldots\ldots\ldots(30)$$

will be equal to those in (5), in a certain order, each to each, because either series gives us the roots of the same biquadratic. But, exactly as it was proved in Prop. v. that Y could not be equal to any of the three first terms of (17), and in Prop. vii. that Y could not be equal to any of the three first terms of (23), it can be proved here that Y is not equal to any of the three first terms of (30). Therefore

$$Y = a_4{}^5 Y_2^4.$$

And this again [by the same reasoning by which it was inferred from equations (24), in Prop. vii., that the expression $B^2 - C$, or YY_1 was the fifth power of a rational quantity] leads to the conclusion that (in the present Proposition) YY_2 is the fifth power of a quantity which is clear of the surd $\sqrt{\{H + K\sqrt{(U)}\}}$, and which only involves the surd $\sqrt{(U)}$. We may put then

$$YY_2 = M^2 = \{e + f\sqrt{(U)}\}^5,$$

and

$$N^2 = \{e - f\sqrt{(U)}\}^5,$$

where e and f are rational. Therefore

$$Y^{\frac{1}{5}} + A_4 Y^{\frac{4}{5}} = Y^{\frac{1}{5}} + Y_2^{\frac{1}{5}} = 2\sqrt{\{e + f\sqrt{(U)}\}}\cos\frac{\theta}{5};$$

and similarly, $A_2 Y^{\frac{2}{5}} + A_3 Y^{\frac{3}{5}} = 2\sqrt{\{e - f\sqrt{(U)}\}}\cos\frac{\phi}{5}$.

Therefore $x_1 = 2\sqrt{\{e + f\sqrt{(U)}\}}\cos\frac{\theta}{5} + 2\sqrt{\{e - f\sqrt{(U)}\}}\cos\frac{\phi}{5}$.

This is the expression for x_1 in (8). Also, putting

$$b^2 = e + f\sqrt{(U)}, \text{ and } \beta^2 = e - f\sqrt{(U)},$$

we have, by adding all the equations (29) together,

$$b^5 \cos\theta + \beta^5 \cos\phi = 2F.$$

Therefore the first expression in (10) is rational. Again, just as Y_2 is equal to a term in (5), which was proved to be the last, each of the terms Y_1 and Y_3 must be equal to a term in (5). The terms in (5) to which Y_1 and Y_3 are severally equal, are of necessity the second and third, because Y_1 and Y_3 can neither be equal to one another, nor to the terms Y, Y_2. As the root $\sqrt{\{H - K\sqrt{(U)}\}}$ may be supposed to have either sign involved in it, we may understand Y_1 to be equal to whichever of the terms $A_2{}^5 Y^2$, $A_3{}^5 Y^3$ we please. Let us put

$$Y_1 = A_3{}^5 Y^3,\ \ Y_3 = A_2{}^5 Y^2 \ldots\ldots\ldots\ldots\ldots(31),$$

and

$$Y_1 = a_2{}^5 Y_2{}^2,\ \ Y_3 = a_3{}^5 Y_2{}^3 \ldots\ldots\ldots\ldots\ldots(32),$$

the two latter equations being the consequence of assuming the former. From the first of equations (31), and the second of (32), we get

$$Y^2 Y_1 = A_3{}^5 Y^5, \text{ and } Y_3 Y_2{}^2 = a_3{}^5 Y_2{}^5;$$

or $\quad M^2 N\{\cos(2\theta + \phi) + \sqrt{(-1)}\sin(2\theta + \phi)\} = A_3{}^5 Y^5,$

and $\quad M^2 N\{\cos(2\theta + \phi) - \sqrt{(-1)}\sin(2\theta + \phi)\} = a_3{}^5 Y_2{}^5.$

Therefore $(M^2 N)^{\frac{1}{5}}\cos\dfrac{2\theta + \phi}{5} = \frac{1}{2}(A_3 Y + a_3 Y_2) = P + R\sqrt{(U)}$;

where P and R are rational expressions. But in like manner it can be proved that

$$(N^2 M)^{\frac{1}{5}}\cos\frac{\theta - 2\phi}{5} = P - R\sqrt{(U)}.$$

And $\quad (M^2 N)^{\frac{1}{5}} = b^2\beta, \text{ and } (N^2 M)^{\frac{1}{5}} = \beta^2 b.$

Therefore $\quad b^2\beta\cos\dfrac{2\theta + \phi}{5} + \beta^2 b\cos\dfrac{\theta - 2\phi}{5} = 2P;$

so that the second expression in (10) is rational. Still further, from the second of equations (31), and the first of (32), we get $Y_3Y^3 = A_2^{\,5}Y^5$, and $Y_1Y_2^{\,3} = a_2^{\,5}Y_2^{\,5}$; which lead to the result

$$b^3\beta\cos\frac{3\theta-\phi}{5} = \tfrac{1}{2}\{P' + R'\sqrt{(U)}\};$$

R' and P' being rational. But in like manner,

$$\beta^3 b\cos\frac{\theta+3\phi}{5} = \tfrac{1}{2}\{P' - R'\sqrt{(U)}\}.$$

Therefore $$b^3\beta\cos\frac{3\theta-\phi}{5} + \beta^3 b\cos\frac{\theta+3\phi}{5} = P';$$

so that the third expression in (10) is rational. Finally, the fourth expression in (10) is the sum of the two expressions

$$\left.\begin{array}{l} 2(b^2+\beta^2)\left\{b^2\beta\cos\dfrac{2\theta+\phi}{5} + \beta^2 b\cos\dfrac{\theta-2\phi}{5}\right\}, \\ \qquad b^2\beta^2\left\{\beta\cos\dfrac{2\theta+\phi}{5} + b\cos\dfrac{\theta-2\phi}{5}\right\}. \end{array}\right\}\ldots(33).$$

In the former of these, the factor $2(b^2+\beta^2)$ is the rational quantity $4e$. The remaining factor is likewise rational, being the second of the expressions in (10). In the latter of the expressions (33) the factor $b^2\beta^2$ is rational. In order therefore that the fourth expression in (10) may be rational, it is only necessary that the expression

$$\beta\cos\frac{2\theta+\phi}{5} + b\cos\frac{\theta-2\phi}{5}$$

be rational. Now, as was proved above,

$$b^2\beta\cos\frac{2\theta+\phi}{5} = \{e + f\sqrt{(U)}\}\,\beta\cos\frac{2\theta+\phi}{5} = P + R\sqrt{(U)},$$

and $$\{e - f\sqrt{(U)}\}\,b\cos\frac{\theta-2\phi}{5} = P - R\sqrt{(U)};$$

therefore

$$\beta\cos\frac{2\theta+\phi}{5} + b\cos\frac{\theta-2\phi}{5} = \frac{P+R\sqrt{(U)}}{e+f\sqrt{(U)}} + \frac{P-R\sqrt{(U)}}{e-f\sqrt{(U)}};$$

where the expression on the right-hand side of the equation is plainly rational. Hence, on the supposition of the proposition, the quintic (3) belongs to what we have termed Class Third.

Cor. As the biquadratic whose roots are the terms in (5) must fall under one or other of the cases considered in the present and preceding propositions, the forms of the roots of the quintic (3) in all the solvible cases have been determined.

In a subsequent number of the *Journal* a brief list will be given of quintic equations belonging to Class Third.

Note.—A few slight errata having crept into the Article Vol. IV., pp. 341–361, on "The Exact Resolution of Algebraical Equations in all the Solvible Cases." On page 351, in the second line of Cor. 1, the word *final* should be inserted after *irreducible*. On the same page, in first line of Cor. 3, the words *and in a simple form*, should be added after *in Cor.* 2. And on page 352, line second, $Y^{\lambda}T^{\lambda_1}$ should be written, as in line 14, for $X^{\lambda}X^{\lambda_1}$. It may also be explained, to prevent mistake, that on page 351, second line of Cor. 2, the phrase *the lowest possible*, which is perhaps ambiguous, means *the lowest which can possibly occur in a final irreducible factor.* G. P. Y.

ON THE CONSTRUCTION OF THE NINTH POINT OF INTERSECTION OF THE CUBICS WHICH PASS THROUGH EIGHT GIVEN POINTS.

By A. Cayley.

I reproduce with additional developments the solution which has been given of this interesting problem.

The equation of a given cubic may be written

$$qU-pV=0,$$

where $U=0$, $V=0$ are any two conics meeting the cubic in the same four points; $p=0$ is the line joining the remaining two points of intersection of the cubic with the conic $U=0$; and $q=0$ is the line joining the remaining two points of intersection of the cubic with the conic $V=0$; the relation between the arbitrary constant factors implicitly contained in the functions qU and pV is assumed to be properly determined.

The form is employed by Plücker, "Theorie der algebraischen Curven," p. 56 (1839), and in connexion therewith he gives some geometrical considerations which, he remarks, contain implicitly the solution of the above mentioned problem.

The form is also the analytical basis of the investigations of M. Chasles, "Construction de la courbe du troisieme ordre par neuf points," *Comptes Rendus*, t. XXXVI. (1853), pp. 942–952. In fact calling to mind the theorem that in the pencil of conics $U-\alpha V=0$ (where α is an arbitrary multiplier) the anharmonic ratio of the multipliers α is equal to the anharmonic ratio of the tangents at any one of the points of intersection, or (what is the same thing) to the anharmonic ratio of the polars of any point whatever in regard to the conics, and recollecting that the anharmonic ratio in question is said to be the anharmonic ratio of the conic themselves; then since the equation $qU-pV=0$ is satisfied by the system

$$U-\alpha V=0,\ p-\alpha q=0,$$

(where α is arbitrary) we have at once the theorem, p. 949, viz. that if there be a pencil of lines through a *point*, and corresponding anharmonically thereto, a pencil of conics through the same *four points;* the locus of the intersections of a line by the corresponding conic is a cubic through the five points; and, conversely, that a given cubic may be so generated, the point of the pencil of lines, and the four points of the pencil of conics being any five points whatever of the cubic.

This gives at once the construction (M. Chasles *first* construction) for the cubic through nine given points. In fact if the points are called 1, 2, 3, 4, 5, 6, 7, 8, 9; then grouping the points in any manner, it is only necessary to find a point x such that

$$x\,(1,\,2,\,3,\,4,\,5)=6789\,(1,\,2,\,3,\,4,\,5),$$

that is, such that the pencil of lines $x1$, $x2$, $x3$, $x4$, $x5$ shall correspond anharmonically to the pencil of conics 67891, 67892, 67893, 67894, 67895. The foregoing notation is that employed in M. de Jonquières' "Essai sur la generation des Courbes geometriques, &c." *Mem. Sav. Etrang.*, t. XVI. (1858), which I take the opportunity of referring to. In fact, if x satisfies the foregoing condition, then taking through the point x any other line, and corresponding anharmonically thereto a conic through the points 6, 7, 8, 9 the locus of the intersections of the line and conic will be a cubic through the nine points. But the condition in question gives

$$x\,(1,\,2,\,3,\,4)=6789\,(1,\,2,\,3,\,4),$$

$$x\,(1,\,2,\,3,\,5)=6789\,(1,\,2,\,3,\,5).$$

which (by the anharmonic property of the points of a conic) show, the first that x is in a certain conic passing through 1, 2, 3, 4, and the second that x is in a certain conic passing through 1, 2, 3, 5; the two conics intersect in the points 1, 2, 3, and in a fourth point which is the required point x. Or we may say that x is given by the condition that the pencils $x(1, 2, 3, 4)$ and $x(1, 2, 3, 5)$ shall have given anharmonic ratios. It will presently be seen how x can be determined by the ruler alone.

Suppose now that the points 1, 2, 3, 4, 5, 6, 7, 8, 9 are the points of intersection of two cubics, the construction should become indeterminate; this is only the case when the two conics which by their intersection should determine x become one and the same conic. This implies that the conic $x(1, 2, 3, 4)$ passes through 5, or that x, 1, 2, 3, 4, 5 are points of the same conic. And then since by the anharmonic property of the points of a conic $x(1, 2, 3, 4) = 5(1, 2, 3, 4)$, we have

$$5(1, 2, 3, 4) = 6789(1, 2, 3, 4).$$

The grouping of the nine points is altogether arbitrary, hence there are in all $(9 \times 70 =)$ 630 such equations, which are really equivalent to only two equations, and which when eight of the points are given, determine the ninth point. Supposing that the given points are 1, 2, 3, 4, 5, 6, 7, 8, the equations for the determination of the remaining point 9 may be taken to be

$$9(5, 6, 7, 8) = 1234(5, 6, 7, 8),$$

$$9(4, 6, 7, 8) = 1235(4, 6, 7, 8),$$

which (it is to be remarked) determine 9 in a similar way to that in which x is given in the construction of the cubic through nine points; viz. 9 is the fourth intersection of two conics which pass through the points 5, 6, 7, 8, and the points 4, 6, 7, 8 respectively. Or we may say that 9 is given by the conditions that the pencils $9(5, 6, 7, 8)$ and $9(4, 6, 7, 8)$ shall have given anharmonic ratios.

The foregoing equations

$$9(5, 6, 7, 8) = 1234(5, 6, 7, 8),$$

$$9(4, 6, 7, 8) = 1235(4, 6, 7, 8),$$

are equivalent to and constitute the geometrical interpretation of the equations obtained (previous to M. Chasles'

Memoir) by Weddle in the paper "On the construction of the ninth point of intersection of two curves of the third degree when the other eight points are given," *Cambridge and Dublin and Mathematical Journal*, t. VI., pp. 83–86 (1851). In fact, reproducing his analysis with only a slight change of notation, let 012 denote the determinant

$$\begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

so that $012=0$ is the equation of the line through the points 1 and 2; and in like manner let 012345 denote the determinant

$$\begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ x_1^2, & \&c. & & & & \\ \vdots & & & & & \end{vmatrix},$$

so that $012345=0$ is the equation of the conic through the points 1, 2, 3, 4, 5. Of course 123, 123456, &c. will denote given functions of the coordinates of the points 1, 2, 3, the points 1, 2, 3, 4, 5, 6, &c. This being so

$$012345.078=\lambda.012347.058$$

is the equation of a particular cubic passing through the points 1, 2, 3, 4, 5, 7, 8, and which if we properly determine λ, viz. if we write

$$\lambda=\frac{612345.678}{612347.658}$$

will also pass through the point 6.

And similarly

$$012345.076=\mu.012347.056$$

is the equation of a particular cubic curve passing through the points 1, 2, 3, 4, 5, 6, 7, and which if we properly determine μ, viz. if we write

$$\mu=\frac{812345.876}{812347.856}$$

will also pass through the point 8. Hence the two curves, each of them passing through the points 1, 2, 3, 4, 5, 6, 7, 8 will intersect in the remaining point 9; and writing 9 for 0, and combining the two equations, we have

$$\frac{978.956}{958.976}=\frac{\lambda}{\mu}=\frac{612345}{612347}\,\frac{812347}{812345},$$

or, what is the same thing,

$$\frac{956.978}{958.967}=\frac{123456.123478}{123458.123467},$$

which is Weddle's equation, and is equivalent to the above mentioned equation

$$9\,(5,\,6,\,7,\,8)=1234\,(5,\,6,\,7,\,8).$$

To prove this I remark that we have identically

$$012.034.514.523-014.023.512.534=012345.$$

In fact the left-hand side equated to zero is the equation of the conic through 1, 2, 3, 4, 5, and such left-hand side must therefore, save to a mere numerical factor, be equal to 012345. And to determine this factor it is to be observed that 012345 contains the term

$$+\,x_0^2.y_1^2.z_2^2.y_3z_3.z_4x_4.x_5y_5,$$

but that there is no such term in 012.034.514.523, and that there is in $-\,014.023.512.534$ the equivalent term

$$-\,x_0y_1z_4.-x_0y_3z_2.x_5y_1z_2.y_3z_3x_4,$$

so that the numerical factor is rightly determined.

The foregoing identity written under the form

$$\frac{012.034}{014.023}-\frac{512.534}{514.523}=\frac{012345}{014.023.514.523}$$

shows that when $012345=0$, i.e. if 0 be a point of the conic through 1, 2, 3, 4, 5, then that we have

$$0\,(1,\,2,\,3,\,4)=5\,(1,\,2,\,3,\,4),$$

which is in fact the anharmonic property of the points of a conic. And observing that $012345=051234$, and substituting 5, 6, for 0, 5 respectively, the identity becomes

$$\frac{512.534}{514.523}-\frac{612.634}{614.623}=\frac{561234}{514.523.614.623}.$$

The equation

$$012345=0$$

may be written

$$012.034-\frac{512.534}{514.523}\,014.023=0,$$

and hence the anharmonic ratio of the conics

$$012345=0,$$
$$012346=0,$$
$$012347=0,$$
$$012348=0$$

is equal to that of the quantities

$$\frac{512.534}{514.523},\quad \frac{612.634}{614.623},\quad \frac{712.734}{714.723},\quad \frac{812.834}{814.823},$$

or calling these quantities for a moment

$$\alpha,\qquad \beta,\qquad \gamma,\qquad \delta,$$

it is

$$=\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\beta-\gamma)},$$

where

$$\alpha-\beta=\frac{512.534}{514.523}-\frac{612.634}{614.623},$$

which is

$$=\frac{561234}{514.523.614.623};$$

and forming in this manner the expressions of each of the four factors $\alpha-\beta$, $\gamma-\delta$, $\alpha-\delta$, $\beta-\gamma$, we have

$$\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\beta-\gamma)}=\frac{561234.781234}{581234.671234},$$

so that in the equation

$$\frac{956.978}{958.967}=\frac{561234.781234}{581234.671234},$$

the right-hand side is

$$=1234\,(5,\,6,\,7,\,8),$$

and since by what precedes the left-hand side is $=9\,(5,6,7,8)$, the equation is

$$9\,(5,\,6,\,7,\,8)=1234\,(5,\,6,\,7,\,8),$$

which is the transformation in question.

Now resuming the two equations

$$9\,(5,\,6,\,7,\,8)=1234\,(5,\,6,\,7,\,8),$$
$$9\,(4,\,6,\,7,\,8)=1235\,(4,\,6,\,7,\,8),$$

the right-hand sides are given anharmonic ratios, and as we have seen the question is to find 9 so that the anharmonic ratios 9 (5, 6, 7, 8), 9 (4, 6, 7, 8) shall have given values. But for the geometrical solution by the ruler alone, we have the preliminary question, from the given eight points, without the assistance of the before mentioned conics, to construct the given anharmonic ratios 1234 (5, 6, 7, 8) and 1235 (4, 6, 7, 8). The solution of both questions is given in Dr. Hart's paper, "Construction by the ruler alone to determine the ninth point of intersection of two curves of the third degree," *Cambridge and Dublin Mathematical Journal*, t. VI., pp. 181, 182 (1851).

The Preliminary Question. The anharmonic ratio 1234 (5, 6, 7, 8) is equal to that of the polars of an arbitrary point X in regard to the conics 12345, 12346, 12347, 12348 respectively (these polars all pass through one and the same point). Now to construct the polars of X in regard to these conics, and first in regard to the conic 12345. The fourth harmonics of X in regard to the lines 12, 34, in regard to 13, 42, and in regard to 14, 23, meet in a point; and considering the several combinations 1234, 1235, 1245, 1345, 2345 we have thus five points; these lie on a line which is the required polar of X in regard to the conic 12345. The polars in regard to the other conics are obtained in the same manner; and it is clear that the first above mentioned point (viz. that deduced from the points 1, 2, 3, 4) is in fact the point of intersection of the four polars, or point of the pencil formed by the polars.

The Principal Question then is, given the points 4, 5, 6, 7, 8 to find the point 9, such that

$$9(5, 6, 7, 8) = 1234(5, 6, 7, 8),$$

$$9(4, 6, 7, 8) = 1235(4, 6, 7, 8),$$

where the right-hand sides represent given anharmonic ratios.

For this, let 65, 74 (fig. 17) meet in M, and on 74 find a point Q such that the anharmonic ratio of the points (4, M, 7, Q) may be equal to the given ratio 1235 (4, 6, 7, 8), say

$$(4, M, 7, Q) = 1235(4, 6, 7, 8),$$

and let 64, 85 meet in N and on 85 find a point R, such that the anharmonic ratio of the points 5, N, R, 8 is equal to the given anharmonic ratio 1234 (5, 6, 7, 8), say

$$(5, N, R, 8) = 1234(5, 6, 7, 8).$$

Join QR meeting 65 in K and 64 in L; then $7K$ and $8L$ will meet in the required point 9.

In fact taking Y as the intersection of the lines $65KM$ and $8L$, and Z as the intersection of the lines $64NL$ and $7K$; then, as is clear from the figure, first, the anharmonic ratio $9(4, 6, 7, 8)$ is equal to that of the point $(4, 6, Z, L)$ on the line $46ZL$, that is

$$9(4, 6, 7, 8) = (4, 6, Z, L),$$

which is $$= (4, M, 7, Q),$$

since the lines $6M$, $Z7$, LQ meet in the point K; but by the construction $(4, M, 7, Q) = 1235(4, 6, 7, 8)$, that is, we have

$$9(4, 6, 7, 8) = 1235(4, 6, 7, 8);$$

and, secondly, the anharmonic ratio $9(5, 6, 7, 8)$ is equal to that of the points $(5, 6, K, Y)$ on the line $56KY$, that is

$$9(5, 6, 7, 8) = (5, 6, K, Y),$$

which is $$= (5, N, R, 8),$$

since the lines $6N$, KR, $Y8$ meet in the same point L; but by the construction $(6, N, R, 8) = 1234(5, 6, 7, 8)$, that is, we have

$$9(5, 6, 7, 8) = 1234(5, 6, 7, 8),$$

so that the point 9 satisfies the required conditions.

It has been already remarked that the point x in M. Chasles' theorem for the construction of the cubic through nine points is determined by precisely similar conditions to those which determine the ninth intersection of the two cubics; that is, the foregoing construction by the ruler alone is applicable to the determination of the point x; and when this is once obtained, the remainder of the construction, giving the point of the cubic through the nine given points, can obviously be performed by the ruler alone. The construction for the cubic through nine points gives implicitly the relation between ten points of the cubic and such relation is accordingly expressed by the equation

$$x(1, 2, 3, 4, 5, 10) = 6789(1, 2, 3, 4, 5, 10),$$

which is one out of 210 similar forms. But it is possible that some more convenient form of the relation between the ten points may yet be found.

I proceed to further develope the analytical theory. Writing for convenience ω in the place of 10, we have

$$x(1, 2, 3, 4, 5, 6) = 789\omega(1, 2, 3, 4, 5, 6),$$

or, what is the same thing,

$$x(1, 2, 3, 4) = 789\omega\,(1, 2, 3, 4),$$
$$x(1, 2, 3, 5) = 789\omega\,(1, 2, 3, 5),$$
$$x(1, 2, 3, 6) = 789\omega\,(1, 2, 3, 6),$$

which belong to three conics each of them passing through 1, 2, 3, and which must have a remaining fourth point of intersection.

The equation of the first conic is

$$012.034 = \frac{\lambda}{\mu}\,014.023,$$

where $$\lambda = 789\omega12.789\omega34,$$
$$\mu = 789\omega14.789\omega23,$$

and thence, in virtue of an identical equation already referred to,

$$-\lambda - \mu = 789\omega13.789\omega24.$$

But we have identically

$$123.0pq = 1pq.023 + 2pq.031 + 3pq.012,$$

and thence in particular

$$123.034 = 134.023 + 234.031,$$
$$123.014 = 214.031 + 314.012,$$

and the equation of the conic may therefore be written

$$012\,(134.023 + 234.031)\,\mu - 023\,(214.031 + 314.012)\,\lambda = 0,$$

that is

$$031.012.234.\mu + 012.023.314\,(-\lambda - \mu) + 023.031.124.\lambda = 0,$$

or, substituting for μ, $-\lambda - \mu$, λ, their values, this is

$$031.012.234.789\omega14.789\omega23$$
$$+\,012.023.314.789\omega13.789\omega42$$
$$+\,023.031.124.789\omega12.789\omega34 = 0,$$

or, what is the same thing,

$$\frac{234.789\omega14.789\omega23}{023} + \frac{314.789\omega13.789\omega24}{031}$$
$$+ \frac{124.789\omega12.789\omega34}{012} = 0,$$

or, making a slight change of form, the equation of the conic is

$$\frac{423.789\omega41.789\omega23}{023}+\frac{431.789\omega42.789\omega31}{031}$$
$$+\frac{412.789\omega43.789\omega12}{012}=0.$$

The equations of the other two conics are deduced by writing successively 5 and 6 in the place of 4; and the condition in order that the conics may have a remaining fourth point of intersection is

$$\begin{vmatrix} 423.789\omega41, & 431.789\omega42, & 412.789\omega43 \\ 523.789\omega51, & 531.789\omega52, & 512.789\omega53 \\ 623.789\omega61, & 631.789\omega62, & 612.789\omega63 \end{vmatrix}=0.$$

This equation, say $\square=0$, expresses the relation between the coordinates of the ten points 1, 2, 3, 4, 5, 6, 7, 8, 9, ω, of the cubic. Hence if 123456789ω denote the determinant

$$\begin{vmatrix} x_1^3, \; y_1^3, \; z_1^3, \; y_1^2z_1, \; z_1^2x_1, \; x_1^2y_1, \; y_1z_1^2, \; z_1x_1^2, \; x_1y_1^2, \; x_1, \; y_1, \; z_1 \\ x_2^3, \text{ \&c.} \\ \vdots \end{vmatrix},$$

123456789ω must be a factor of $\square$, and a little consideration shows that the other factor which is of the order one as regards the coordinates of each of the points 1, 2, 3, and of the order three as regards the coordinates of each of the points 7, 8, 9, ω, must be of the form $123.789.78\omega.79\omega.89\omega$. We must therefore have

$$\square=\varepsilon.123.789.78\omega.79\omega.89\omega.123456789\omega,$$

where the merely numerical factor ε is, I believe, equal to $+1$ or else to -1.

In order to verify the factor $123.789.78\omega.79\omega.89\omega$, observing that the points 7, 8, 9, ω enter symmetrically, it will be sufficient to shew that 123, 789 are each of them factors of $\square$, or, what is the same thing, that if $123=0$, or if $789=0$, then in either case $\square=0$.

First, if $123=0$, we may write

$$x_3=\lambda x_1+\mu x_2,$$
$$y_3=\lambda y_1+\mu y_2,$$
$$z_3=\lambda z_1+\mu z_2$$

equations which give

$$423 = \lambda.421, \quad 431 = \lambda.421, \ \&c.,$$

and the equation $\square = 0$, thus becomes

$$\begin{vmatrix} 789\omega41, & 789\omega42, & 789\omega43 \\ 789\omega51, & 789\omega52, & 789\omega53 \\ 789\omega61, & 789\omega62, & 789\omega63 \end{vmatrix} = 0,$$

or, what is the same thing,

$$\begin{vmatrix} 789\omega14, & 789\omega24, & 789\omega34 \\ 789\omega15, & 789\omega25, & 789\omega35 \\ 789\omega16, & 789\omega27, & 789\omega36 \end{vmatrix} = 0.$$

Now if the terms in the same column are multiplied by $789\omega56$, $789\omega64$, $789\omega45$ respectively and added, then for the first column the sum is

$$789\omega14 . 789\omega56 + 789\omega15 . 789\omega64 + 789\omega16 . 789\omega45,$$

which is $= 0$, and the sums for the second and third columns are each $= 0$ in the same manner: wherefore the determinant vanishes as it should do.

Next, if $789 = 0$, we have identically

$$789\omega41 = 789 . 741 . \omega81 . \omega94 + 781 . 794 . \omega89 . \omega41,$$

which when $789 = 0$ gives

$$789\omega41 = 781 . 794 . \omega89 . \omega41,$$

and in like manner,

$$789\omega42 = 782 . 794 . \omega89 . \omega42,$$

$$789\omega43 = 783 . 794 . \omega89 . \omega43,$$

in which three equations 4 may be changed into 5 and 6 successively. The equation $\square = 0$ thus becomes

$$\begin{vmatrix} 423 . \omega41, & 431 . \omega42, & 412 . \omega43 \\ 523 . \omega51, & 531 . \omega52, & 512 . \omega53 \\ 623 . \omega61, & 631 . \omega62, & 612 . \omega63 \end{vmatrix} = 0,$$

or, what is the same thing,

$$\begin{vmatrix} 423 . 41\omega, & 421 . 4\omega3, & 42\omega . 431 \\ 523 . 51\omega, & 521 . 5\omega3, & 52\omega . 531 \\ 623 . 61\omega, & 621 . 6\omega3, & 69\omega . 631 \end{vmatrix} = 0,$$

and since the sum of the terms in each line of the determinant is $= 0$, the determinant is as it should be $= 0$.

The foregoing equation

$$\begin{vmatrix} 412.789\omega 43, & 423.789\omega 41, & 431.789\omega 42 \\ 512.789\omega 53, & 523.789\omega 51, & 531.789\omega 52 \\ 612.789\omega 63, & 623.789\omega 61, & 631.789\omega 62 \end{vmatrix}$$

$$= \varepsilon.123.789.78\omega.79\omega 89\omega.123456789\omega,$$

(since 412, $789\omega 43$, &c. are interpretable functions of the coordinates) affords a geometrical interpretation of the equation

$$123456789\omega = 0$$

between the coordinates of the ten points of the cubic; but it would be more satisfactory if a similar identical equation could be found, having on the right-hand side the function 123456789ω without the irrelevant factor

$$123.789.78\omega.79\omega.89\omega.$$

2, Stone Buildings, W.C.,
March 6th, 1862.

ON M'CULLAGH'S PROPERTY OF A SELF-CONJUGATE TRIANGLE, AND SIR W. HAMILTON'S LAW OF FORCE FOR A BODY DESCRIBING A CONIC SECTION.

By John Casey, Scholar of Trinity College, Dublin, and Science-Master, Kingstown Schools, Kingstown.

LEMMA I. *Let KH, DI be the polars of the points A, B, (the reader can easily construct the figure) with respect to any conic, C its centre, draw AG, BH parallel to CB, CA, respectively, and intersecting DI, KH in the points G, H. Then the rectangle $AG.CB : AC.BH ::$ square of the semi-diameter through B : square of the semi-diameter through A.*

DEMONSTRATION. Draw BL parallel to KH to meet AC in L, and produce KH to meet CB in M, and AC to meet DG.

Now $CL.CI = CA.CK$, each rectangle being equal to the square of the semi-diameter through A.

Hence $\quad CI : CA :: CK : CL :: CM : CB;$

therefore $\quad CI : AI :: CM : BM,$

and hence by similar triangles,

$$CD : AG :: CK : BH;$$

therefore $AG.CB : BH.AC :: CD.CB : CK.AC,$

that is :: square of the semi-diameter through B: square of semi-diameter through A. Q. E. D.

Cor. This theorem becomes simplified in the case of the parabola. For the semi-diameters through the points B and A are equal and parallel since the centre is at infinity, and so also are CA and CB, hence $AG = BH$.

Lemma II. *In any conic section X (see fig. 18). If MN be the polar of a given point A, and P any point on the curve, AF a perpendicular on the tangent at P, PG the portion of the normal at P intercepted between the curve and its axis, and PM a perpendicular from P on MN. Then PM varies as the rectangle $PG.AF$.*

Demonstration. Draw PR parallel to AC, and AE to CP, and CL perpendicular to the tangent, and CQ on MN. Then since P is the polar of the tangent at P, and A of MN, we have from the demonstration of Lemma I.,

$$PR : CN :: AE : CP.$$

Hence from the similar triangles PRM, CNQ, and AFE, CLP,

$$PM : CQ :: AF : CL :: AF.PG : CL.PG,$$

but CQ is given, and so is $CL.PG$, being equal to the square of the semi-conjugate diameter, hence the proposition is evident. Q. E. D.

The foregoing proof must be modified a little for the parabola. Thus (fig. 19), by similar triangles,

$$AE : AF :: PG : GN,$$

(PN being perpendicular to the axis) but $AE = PR$, and $GN =$ half the latus rectum, and is given; therefore PR varies as $PG.AF$, and PM evidently varies as PR, hence PM varies as $PG.AF$. Q. E. D.

1. M'Cullagh's Theorem. *If P_1, P_2, P_3 be the perpendiculars from the angles of a given self-conjugate triangle with respect to any given conic on any tangent to the conic, and p_1, p_2, p_3 the perpendiculars from the point of contact on*

the opposite sides of the triangle, then the ratio $\frac{P_1^2}{P_2 P_3} : \frac{p_1^2}{p_2 p_3}$ *is constant.*

This is evident from Lemma II.

The foregoing theorem was discovered by the late lamented Professor M'Cullagh of Trinity College, Dublin. He arrived at it from the known equation of a conic referred to the three sides of a self-conjugate triangle in a system of trilinear coordinates.

2. Sir W. Hamilton's Law of Force. *If X be any given conic, O any point in its plane, F a central force at O which causes a material particle to describe X, then F varies as* $\frac{r}{p^3}$, *where r is the radius vector from O to the particle, and p the perpendicular from the particle to the polar of O.*

Demonstration. Let P be the perpendicular from O on the tangent to the conic through the particle, and R the radius of curvature. Then by De Moivre's law of force, $F = \frac{h^2}{P^3}\frac{dP}{dr}$, and $\frac{dP}{dr} = \frac{r}{R}$ (Todhunter's *Differential Calculus*, Art. 323); therefore $F = \frac{h^2 r}{P^3 R}$. Now in any conic R varies as the cube of the normal, hence by Lemma II. the proposition is evident. Q. E. D.

The following are particular cases of the foregoing proposition:

1°. *When the point O is at the centre F varies directly as the radius vector.*

2°. *When O coincides with a focus, F varies inversely as the square of the radius vector.*

3°. *When O is at infinity, F varies inversely as the cube of the ordinate from the particle to the diameter conjugate to the line from O the centre.*

Hence also the path of a particle attracted towards an infinite plane according to the inverse cube of its distance from the plane is a conic section.

Model School, Kilkenny,
July, 16, 1861.

ON A PROPERTY OF THE FUNDAMENTAL TRIANGLE IN TRILINEAR COORDINATES.

By J. CORBETT TURNBULL, M.A., Trinity College, Cambridge, and Head Mathematical Master of Cheltenham College.

AD is the perpendicular from A on BC (fig. 20). DE, DF perpendicular to AC, AB. Join FE, and produce it to meet BC in G.

Let $FE=a'$, and let b', c' be corresponding lines for the other perpendiculars of the triangle.

Then if H, K be their intersections with b, c respectively, G, H, K will lie in the line whose equation is

$$\alpha \operatorname{cosec} A + \beta \operatorname{cosec} B + \gamma \operatorname{cosec} C = \frac{(2\Delta)^2}{abc}.$$

From similar triangles AFE, ABC,

$$\frac{a'}{a} = \frac{AF}{b} = \frac{AD \sin B}{b};$$

therefore $$a' = 2\Delta . \frac{\sin A}{a}.$$

So $$b' = 2\Delta . \frac{\sin B}{b},$$

$$c' = 2\Delta . \frac{\sin C}{c};$$

therefore $$a' = b' = c;$$

therefore $$a'^3 = \frac{(2\Delta)^6}{(abc)^3};$$

therefore $$a' = \frac{(2\Delta)^2}{abc}.$$

Now $\beta \operatorname{cosec} B + \gamma \operatorname{cosec} C = a'$ is the equation to EF,

$\alpha \operatorname{cosec} A + \gamma \operatorname{cosec} C = b'$ .... b',

$\alpha \operatorname{cosec} A + \beta \operatorname{cosec} B = c'$ .... c';

therefore the points of intersection of a', b', c' with a, b, c respectively, will lie in the line

$$\alpha \operatorname{cosec} A + \beta \operatorname{cosec} B + \gamma \operatorname{cosec} C = \frac{(2\Delta)^2}{abc}.$$

April, 1862.

ON ABELIAN CUBICS AND ON SYMMETRICAL EQUATIONS.

By JAMES COCKLE.

APART from Mr. Jerrard's criticism of Legendre, there has been a conflict of authority on the subject of cubic and other abelians to which I have elsewhere* referred. And a passage of my concluding "Notes on the Higher Algebra" leaves the point touched on by Mr. Jerrard still undecided, for I have improperly assigned as a unique form that which is merely a variety of a particular species. The equation (*Journal* for October, 1861, p. 3)

$$x_2^2 + (x_1 + a)\, x_2 + x_1^2 + ax_1 + b = 0$$

is susceptible of transformations of which the following is one. Multiply it into x_1^2: the result, after easy substitutions and a change of its sign, may be made to take the form

$$(ax_1^2 + bx_1 + c)\, x_2^2 + (bx_1^2 + cx_1)\, x_2 + cx_1^2 = 0,$$

whence, after proper reductions,

$$x_2 = \frac{c + bx_1}{2x_1^2} \mp \frac{\sqrt{\{(b^2 - 4ac)\, x_1^2 - 2bcx_1 - 3c^2\}}}{2x_1^2}.$$

Now, in order that the radical may become rational, we have the condition

$$-12c^2 (b^2 - 4ac) = 4b^2c^2,$$

or

$$16c^2 (b^2 - 3ac) = 0,$$

or, rejecting $c^2 = 0$,

$$b^2 - 3ac = 0.$$

If this condition be satisfied the cubic is an abelian, and we have, moreover,

$$x_2 = \frac{c + bx_1}{2x_1^2} \mp \frac{3c + bx_1}{2x_1^2 \sqrt{(-3)}}$$

$$= -\frac{a + x_1}{2} \pm \frac{3(a + x_1)}{2\sqrt{(-3)}} \pm \frac{b}{x_1 \sqrt{(-3)}}.$$

* *Philosophical Magazine* for December, 1849, pp. 437, 438; for June 1852, pp. 457—460; and see Art. (90) pp. 327, 8, and Arts. (302) and (303) pp. 381, 382 of Mr. De Morgan's "Calculus of Functions" in the *Encyclopædia Metropolitana*, Vol. II. of the Pure Sciences.

Let us for convenience put

$$\omega=-\frac{1}{2}+\frac{\sqrt{(-3)}}{2},\ \omega^2=-\frac{1}{2}-\frac{\sqrt{(-3)}}{2},$$

ω and ω^2 being the unreal cube roots of unity. Also let us write

$$\theta x=\omega(a+x)-\frac{b}{x\sqrt{(-3)}},$$

$$\theta^2 x=\omega^2(a+x)+\frac{b}{x\sqrt{(-3)}},$$

and apply Abel's process. We have

$$\begin{aligned}\psi x&=(x+\omega\theta x+\omega^2\theta^2 x)^3\\&=\left\{(\omega+\omega^2)a-(\omega-\omega^2)\frac{b}{x\sqrt{(-3)}}\right\}^3\\&=-\left(a+\frac{b}{x}\right)^3;\\\Psi x&=(x+\omega^2\theta x+\omega\theta^2 x)^3\\&=\left\{3x+2a+(\omega-\omega^2)\frac{b}{x\sqrt{(-3)}}\right\}^3\\&=\left(3x+2a+\frac{b}{x}\right)^3=(3x+a-\psi x)^3;\end{aligned}$$

and following Abel's track, we should next find

$$\psi x=\frac{1}{3}\Sigma\psi x,\ \Psi x=\tfrac{1}{3}\Sigma\Psi x,$$

the three roots x_1, x_2, and x_3 being successively inserted in the expressions which follow Σ. But there is a shorter route in the present case. Putting the given cubic under the reciprocal form

$$\frac{1}{x^3}+\frac{b}{c}\cdot\frac{1}{x^2}+\frac{a}{c}\cdot\frac{1}{x}+\frac{1}{c}=0,$$

multiplying it into b^3, and introducing the relation

$$b^2-3ac=0,$$

we can reduce it to

$$\left(\frac{b}{x}+a\right)^3+3ab-a^3=0,$$

which determines ψx thus;—

$$\psi x = a^3 - 3ab.$$

Further,

$$\Psi x = \left(\frac{3x^3 + 2ax^2 + bx}{x^2}\right)^3$$

$$= -\left(\frac{a^2x^2 + 2abx + b^2}{ax^2}\right)^3 = -\frac{1}{a^3}\{(\psi x)^2\}^3,$$

and
$$x = -\frac{1}{3}\{a - \sqrt[3]{(\psi x)} - \sqrt[3]{(\Psi x)}\}$$

$$= -\frac{1}{3}\left[a + \sqrt[3]{(a^3 - 3ab)} + \frac{1}{a}\sqrt[3]{\{(a^3 - 3ab)^2\}}\right].$$

This second variety of what we may call the first species of abelian cubic is identical with the solvible form which I gave in the *Cambridge Mathematical Journal** for May, 1841. For the first variety (in which $a^2 - 3b = 0$) we have

$$\theta x = \omega(a + x) + \frac{a}{\sqrt{(-3)}}, \quad \theta^2 x = \omega^2(a + x) - \frac{a}{\sqrt{(-3)}},$$

$$\psi x = \left\{(\omega + \omega^2)a + (\omega - \omega^2)\frac{a}{\sqrt{(-3)}}\right\}^3 = 0,$$

$$\Psi x = \left\{3x + 2a - (\omega - \omega^2)\frac{a}{\sqrt{(-3)}}\right\}^3$$

$$= (3x + a)^3 = a^3 - 27c.$$

On referring to the reciprocal equation we see that the condition of its being an abelian of the first variety is

$$\left(\frac{b}{c}\right)^2 = 3\left(\frac{a}{c}\right), \quad \text{or } b^2 = 3ac.$$

Hence the cubic whose roots are the reciprocals of an abelian of the first variety is an abelian of the second variety; and the cubic whose roots are the reciprocals of an abelian of the second variety is an abelian of the first variety. In the cases in which the first or the last coefficient of the Hessian vanishes we have abelian cubics of the first or the second variety respectively.

A second species of abelian cubic is that considered by M. Serret in *Note* VII. of his *Cours*, 2^{me}. ed. M. Serret's

* See my third and concluding series of Notes on the Theory of Algebraic Equations in Vols. LII.—LV. of the *Mechanics' Magazine*.

researches are based upon those of M. Lobatto, which are given in the *Seizième Leçon* of the *Cours*. The researches of M. Lobatto were, in their turn, founded upon, or suggested by,* those of James Lockhart and Professor J. R. Young. The property which gave rise to the discussion is, that two roots of a cubic can always be expressed rationally in terms of the third root. Every cubic is therefore, in one sense, an abelian; but the difficulty is, that the obtaining the actual value of the rational function, and indeed the demonstration that such a function exists, involves the solution of the given cubic. And the investigations of Weddle and of Hearn† are, in strictness, only demonstrations of Lockhart's‡ theorem. The discussion which follows shows *a priori*, and without any assumption as to the solution of the given cubic, that any root of any cubic is a rational function of another root, and it moreover furnishes the means of ascertaining the precise form of the function.

I shall first show that the square root of any linear function whatever of a root of a quadratic or a cubic can be expressed as a rational function of that root. This theorem is a branch of the more general inquiry, as to when it is possible to express a function

$$\sqrt[m]{(\psi x)},$$

ψx be a rational function of a root of an equation of any degree, as a rational function of that root.

Let $$fx = x^2 + ax + b = 0$$

be the quadratic, and $m(x+\alpha)$ a linear function of x, m and α being constants. Then

$$\sqrt{\{m(x+\alpha)\}} = \sqrt{\{m(x+\alpha) + \lambda fx\}}$$
$$= \sqrt{\{\lambda x^2 + (\lambda a + m)x + \lambda b + m\alpha\}},$$

and the expressions under the radical sign will be a square, provided that

$$4\lambda(\lambda b + m\alpha) = (\lambda a + m)^2,$$

or $$(a^2 - 4b)\lambda^2 + 2m(a - 2\alpha)\lambda + m^2 = 0.$$

* See the *Note sur une propriété relative aux racines d'une classe particulière d'equations du troisième degré*, par M. Lobatto, Professeur de Mathématiques à l'Académie royale de Delft, printed in *Liouville*, for May, 1844; t. IX. pp. 177—190.

† See *Mathematician*, for November, 1845, Vol. II., pp. 43, 44.

‡ See *Mathematician*, for July, 1845, Vol. I., p. 334; for November, 1845, Vol. II. p. 42.

Determinnig λ from this quadratic, we have

$$\sqrt{\{m(x+a)\}} = \lambda^{\frac{1}{2}}x + \tfrac{1}{2}(\lambda^{\frac{1}{2}}a + \lambda^{-\frac{1}{2}}m),$$

the dexter of which is a rational function of x. It may be noticed that any rational function of x and its powers, x being the root of a quadratic, may be put under the form $m(x+a)$.

Perhaps the more general inquiry might not be unprofitable, but I have as yet confined myself to the case of the cubic, which yields the following result.

From $fx=0$ we readily find

$$-x = \frac{ax^2+bx+c}{x^2}$$

$$= \frac{b^2}{4c} + \frac{b}{x} + \frac{c}{x^2} + a - \frac{b^2}{4c}$$

$$= \left(\frac{b}{2\sqrt{c}} + \frac{\sqrt{c}}{x}\right)^2 + a - \frac{b^2}{4c},$$

or putting

$$\frac{b}{2\sqrt{c}} + \frac{\sqrt{c}}{x} = H,$$

$$-x = H^2 + a - \frac{b^2}{4c}.$$

Consequently

$$\sqrt{(-x)} = \sqrt{\left(H^2 + a - \frac{b^2}{4c}\right)},$$

$$\sqrt{(K-x)} = \sqrt{\left(H^2 + K + a - \frac{b^2}{4c}\right)}.$$

Let

$$K = 2Hh + h^2 - a + \frac{b^2}{4c},$$

and the preceding equation becomes

$$\sqrt{\left(h^2 + 2Hh - a + \frac{b^2}{4c} - x\right)} = \pm(H+h) \ldots\ldots (A).$$

Now H is a rational function of x, and the arbitrary quantity h is to be a rational function of x so determined that the expression under the radical shall be a linear function of x. But this last equation may be put under the form

$$\sqrt{\left\{h^2x^2 - 2Hxhx - \left(a - \frac{b^2}{4c}\right)x^2 - x^3\right\}} = \pm(H+h)\,x \ldots\ldots (B),$$

where the expression under the radical becomes, on reduction,

$$(hx)^2+\left(\frac{bx}{\sqrt{c}}+2\sqrt{c}\right)hx-\left(a-\frac{b^2}{4c}\right)x^2-x^3,$$

or $$(hx)^2+\left(\frac{bx}{\sqrt{c}}+2\sqrt{c}\right)hx+\frac{b^2}{4c}x^2-ax^2-x^3.$$

Assume

$$h=\beta x+\gamma,\quad hx=x^2\left(\beta+\frac{\gamma}{x}\right),$$

then the last preceding quantity becomes, when developed and divided by x^2,

$$\left(\beta^2-\frac{2\gamma}{\sqrt{c}}\right)x^2+\sqrt{c^{-1}}\,\{2\beta\gamma\sqrt{c}-2a\gamma+\beta b-\sqrt{(c)}\}\,x+\gamma^2-a$$

$$+\frac{b^2}{4c}-\frac{b\gamma}{\sqrt{c}}+2\beta\sqrt{c},$$

or, if $$-\beta^2+\frac{2\gamma}{\sqrt{c}}=0,\quad i.e.\ \gamma=\frac{\beta^2\sqrt{c}}{2},$$

it becomes

$$\sqrt{(c^{-1})}\,\{2\beta\gamma\sqrt{(c)}-2\gamma a+\beta b-\sqrt{(c)}\}\,x+\gamma^2-a+\frac{b^2}{4c}+2\beta\sqrt{c}-\frac{b}{\sqrt{c}}\gamma,$$

which we may write $\qquad Px+Q.$

Hence if we make

$$\frac{Q}{P}=\frac{\gamma^2\sqrt{c}-a\sqrt{c}+\dfrac{b^2}{4\sqrt{c}}+2\beta c-b\gamma}{2\beta\gamma\sqrt{c}-2a\gamma+b\beta-\sqrt{c}}=\alpha,$$

where α is arbitrary, and determine γ from this biquadratic, we arrive finally at the relation

$$\sqrt{(x+\alpha)}=\frac{\pm(H+h)}{\sqrt{(P)}};$$

in other words we have expressed the radical $\sqrt{(x+\alpha)}$ as a rational function of x, and of course the square root of any other linear function of x, *ex. gr.*, $\sqrt{\{m(x+\alpha)\}}$, where m and α are arbitrary constants, can be so expressed.

In developing the expression under the radical sign in the sinister of (B), the operation is not lengthy. For, remembering that the result is to be divided by x^2, it will be found that $\frac{1}{x}$ is the only power of x that will have to

be eliminated; and this will only happen once, if the operation be properly conducted. Thus in the substitutions given by the relations

$$(hx)^2 = x^2(\beta^2x^2 + 2\beta\gamma x + \gamma^2),$$

$$2Hx.hx = \left(\frac{bx}{\sqrt{c}} + 2\sqrt{c}\right)x^2\left(\beta + \frac{\gamma}{x}\right),$$

the x^2 is to remain untouched, and ultimately to be divided out. The final x^2 of the surd expression must also remain untouched: for if a substitution be made for it in terms of x^2, x, a, &c., we shall be led to a result of the form

$$\alpha = \frac{Q}{P} = \frac{0}{P}.$$

This illusory result arises from the substitution in question bringing back the quantity under the radical to a perfect square. On the other hand, if (which is the most direct, and probably the simplest, process) we substitute $\beta x + \gamma$ for h, in the expression under the radical sign in the sinister of (A), then x^2 may be eliminated without giving rise to an illusory result. In either case we are led to the same expression for α, viz. that given above, which, substitution being made for γ in terms of β^2, becomes

$$\frac{Q}{P} = \frac{c^2\beta^4 + 8c^{\frac{3}{2}}\beta - 2bc\beta^2 + b^2 - 4ac}{4(c^2\beta^3 + bc^{\frac{1}{2}}\beta - ac\beta^2 - c)} = \alpha.$$

The value of α thus arrived at is not in general illusory, but β is determined as the root of a biquadratic. The solution of the problem before us does not however require that of any subsidiary equation higher than a quadratic, for if we assume

$$h = \vartheta x^2 + \beta x + \gamma,$$

and substitute this value for h in the sinister of (A), the surd expression may be put under the form

$$\sqrt{(Mx^2 + Px + Q)},$$

in which M, P and Q are, in general, quadratic, though not homogeneous, functions of ϑ, β and γ. Hence, by that general process which I have termed the "Method of Vanishing Groups," we may, without having recourse to Mr. Jerrard's Method of Decomposition, so determine γ that

$$M = 0$$

shall be satisfied by means of a linear relation (in which A and B are known quantities)

$$\vartheta - A\beta - B = 0$$

between ϑ and β. Hence, eliminating ϑ, we shall be led to

$$\alpha = \frac{Q}{P},$$

in which P and Q are quadratic functions of β; and β will be determined by a quadratic equation.

It follows that the square root of any rational function whatever, ψx, of a root, x, of a cubic, can be expressed as a rational function of x. For let $\psi x = x'$, and form the transformeé in x'. Then, by what precedes, $\sqrt{(x'+\alpha')}$, α' being arbitrary, can be expressed as a rational function of x', and, consequently, of x. Let α' vanish and we have $\sqrt{(x')}$ or $\sqrt{(\psi x)}$ expressed as a rational function of x.

It is not, however, in the case before us, necessary to have recourse to this transformation. For the sake of brevity, let a vanish, or

$$fx = x^3 + bx + c = 0.$$

Then from $$\{fx_2 - fx\} \div (x_2 - x) = 0,$$

we deduce

$$x_2 = -\frac{x}{2} \pm \frac{1}{2}\sqrt{\{-(3x^2+4b)\}},$$

$$= -\frac{x}{2} \pm \frac{1}{2}\frac{\sqrt{\{(3x^2+4b)(x+\alpha)\}}}{\sqrt{\{-(x+\alpha)\}}}.$$

But the numerator of the last term becomes on reduction

$$\sqrt{\{3\alpha x^2 + bx + (4b\alpha - 3c)\}},$$

which is a rational function of x, provided that

$$4.3\alpha(4b\alpha - 3c) + b^2,$$

or $$48b\alpha^2 - 36c\alpha = b^2.$$

Solve this quadratic, and we have

$$\alpha = \frac{3c}{8b} \pm \frac{1}{4}\sqrt{\left(\frac{b}{3} + \frac{9c^2}{4b^2}\right)}$$

$$= \frac{3}{4b}\left\{\frac{c}{2} \pm \sqrt{\left(\frac{b^3}{27} + \frac{c^2}{4}\right)}\right\},$$

and the well known quadratic radical makes its appearance. We have, moreover,

$$x_2 = -\frac{x}{2} \pm \frac{\sqrt{(-1)}}{3a} \cdot \frac{3ax + \frac{b}{2}}{\sqrt{(x+a)}}.$$

But we know that $\sqrt{(x+a)}$ can be expressed as a rational function of x. Therefore x_2 can be expressed as a rational function of x. In other words we have a theorem which seems to dispose of the point to which Mr. De Morgan has called attention, and on which Mr. Jerrard has made observations. The theorem is:—that one of the roots of a general cubic can, without assuming its solution known and without having to solve any equation higher than a quadratic, be expressed as a rational function of another root; and consequently that a general cubic may be solved as an abelian equation. And this result is in accordance with the conclusion of M. Kronecker (see Serret's *Cours*, 2me ed., Note XIII., p. 564), that every equation of a prime degree μ soluble algebraically is an abelian equation, when we regard as known a quantity ρ, which itself is a root of an abelian* equation of the degree $\mu - 1$.

I subjoin in a footnote† another illustration of a process which I have here used, and which may possibly serve an

* It has been noticed by Mr. De Morgan (*Calculus of Functions*, Art. (303), p. 382, col. 2), and it may be readily proved, that every quadratic is an abelian. Let

$$x^2 + ax + b = 0$$

be a quadratic, and x and x_2 its roots. Then

$$x_2 = -(a + x) = \theta x,$$

a rational function of x. And

$$\theta^2 x = -\{a - (a + x)\} = x.$$

Abel's process applied to the above quadratic, and modified as in the text, gives

$$\psi x = (x - \theta x)^2 = (a + 2x_1)^2 = a^2 - 4b,$$

whence
$$x = -\frac{a}{2} \pm \frac{\sqrt{(a^2 - 4b)}}{2}.$$

So, if we take $x_2 = \frac{b}{x} = \theta x$, we have $\theta^2 x = x$, and we might thence deduce the solution.

† Given one of the trinomial forms of Mr. Jerrard

$$x^5 + cx^2 + e = 0,$$

ulterior use in the theory of the surd transcendents of the integral calculus. It has a certain connection with an application of Mr. Gompertz's system of porismatization which I made to the theory of equations* (to the theory of absolutely impossible surd equations), and to the (attempted) geometrical representation of absolutely impossible quantity.† Another and most successful application of Mr. Gompertz's system was subsequently made by Professor J. R. Young‡ to the determination of the nature (real or unreal) of the roots of equations.

ON SYMMETRICAL EQUATIONS.

Let
$$F(x_2, x) = 0$$
be a symmetrical relation connecting x_2 and x. Then as we have seen (*Journal* for October, 1861, pp. 10, 11, footnote †) we may have, in such a case as
$$F(x_2, x) = x_2^3 + x^3 - 1 = 0 \ldots\ldots\ldots\ldots\ldots (1),$$
the result
$$x_2 = \psi x, \quad \psi^2 x = x,$$
which follows from the symmetry of the equation $F(x_2, x) = 0$,

we have
$$\sqrt{(-x)} = \sqrt{\left(\frac{e}{x^4} + \frac{c}{x^2} + \frac{c^2}{4e} - \frac{c^2}{4e}\right)} = \sqrt{\left(H^2 - \frac{c^2}{4e}\right)},$$
$$\sqrt{\left(h^2 + 2Hh + \frac{c^2}{4e} - x\right)} = \pm (H + h) \ldots\ldots\ldots\ldots\ldots (a),$$
where
$$H = \frac{\sqrt{e}}{x^2} + \frac{c}{2\sqrt{e}}.$$
Hence
$$\sqrt{\left\{(hx^2)^2 + 2Hx^2hx^2 + \frac{c^2x^4}{4e} - x\right\}} = \pm (H + h)\, x^2 \ldots\ldots\ldots (b),$$
and if in this case we assume
$$h = F(x),$$
we may seek to proceed as in the instance given in the text, but expressing the square root of a certain function of a root as a rational function of that root.

* *Horæ Algebraicæ.* See *Mechanics' Magazine* for October 23, 1847 (Vol. XLVII., pp. 409, 410), and for February 19, 1848 (Vol. XLVIII., pp. 181—183).

† *Philosophical Magazine* for February, 1849, Ser. 3, Vol. XXXIV., pp. 132—135.

‡ In the "Analysis of Numerical Equations," which is the only one of his Tracts "On the General Principles of Analysis" that has yet appeared.

and which, if it held generally, would be fatal not only to a theory of the (in general) transcendental functions which I have denoted by ϕ, but also to that of abelian equations. We know, however, that, n being prime and greater than 2, no two terms of the series of rational functions

$$x,\ \theta x,\ \theta^2 x,\ \ldots \theta^{n-1} x$$

are equal. And yet we have the symmetrical relation

$$F(\theta x, x) = 0$$

connecting θx and x. The question is how to reconcile the apparently conflicting results.

It is to be observed that when from

$$F(x_2, x) = (fx_2 - fx) \div (x_2 - x) = 0 \ldots\ldots\ldots (2),$$

we have found

$$x_2 = \psi x,$$

the equation

$$F(\psi x, x) = F(x, \psi x) = 0,$$

is satisfied identically, so that any process by which we should seek to deduce, universally, the equation

$$x = \psi\psi x = \psi^2 x$$

would conduct us to a relation of the form

$$x = \frac{0}{0}.$$

It is true that

$$F(x_2, x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

may be supposed to be satisfied by either of the two systems

$$x_2 = \psi x, \quad x = \psi x_2;$$

but we cannot by combining the two (thus:

$$x = \psi x_2 = \psi\psi x_1 = \psi^2 x)$$

conclude that, universally, ψ is a periodic function of the second order. When once we have made the substitution of ψx for x_2, or of ψx_2 for x in (3), x, or x_2, alike disappear from a result, which vanishes identically. And there are no grounds for establishing any such further connection between x_2 and x_1, as the above combination implies. In the cases in which $\psi^2 x = x$, such property can only be known *a posteriori*, and by examining the form of ψ or by reference

to the equation $fx=0$, which, when its degree is even, admits of such a result as $\psi^2 x = x$.

Unless then the degree of the given equation $fx=0$ is even, we cannot in general have $\phi^2 x = x$. I have dwelt upon this point because we have seen (*Journal* for October, 1861, pp. 5, 6) what a tendency there is in the equation $\phi^2 x = x$ (and in another inadmissible equation $\phi^3 x = \phi^2 x$) to obtrude itself improperly into the processes.

4, Pump Court, Temple, London,
November 28, 1861.

ON THE THEORY OF QUINTICS.

Part II.

By the Rev. Robert Harley.

[Part I. will be found at pp. 343—359 of Vol. III.]

From the last three equations in Art. 5, I inferred that β can be expressed as a rational function of t, and the coefficients of the trinomial equation. The inference is erroneous; for t is a 12-valued function, whereas β is a 120-valued function, and even β^5 is a 24-valued function. It hence appears that β cannot be expressed as a rational function of t, and that, consequently, the sextic in t is not, as I was tempted to suppose, an Abelian. Misgivings on this point occurred to me in writing my paper, as will be seen on referring to the last paragraph of Art. 8.

Mr. Cockle had previously committed a similar oversight: see Arts. 58, 59 of his "Observations on the Theory of Equations of the Fifth Degree," published in the *Philosophical Magazine* for July, 1859. Subsequently, however, in the same Journal (March, 1866) he corrected the oversight, and abandoned the idea of the algebraic solvibility of the quintic.

Some Remarks* by Mr. G. B. Jerrard on my former Paper and on Mr. Cockle's "Observations" will be found in the *Philosophical Magazine* for April, 1860, and for January, 1861. Some Notes† in reply by Mr. Cockle appeared in the same Journal. (See Numbers for May, 1860, and February, 1861.) Into the discussion between these two mathematicians, so far as it relates to the validity of the processes employed by Mr. Jerrard, I do not wish at present to enter: I shall confine my attention here to those remarks which bear immediately upon my own Paper.

Assuming that the sextic in t is capable of being solved by Abel's method, Mr. Jerrard notices that since, according to Art. 9 of my Paper,

$$t_1t_3 + t_2t_5 + t_4t_6 = x_1(3^2Q - 2x_1^5),$$

it follows that the roots of the trinomial equation must admit of being expressed by means of quadratic and cubic radicals only. He hence infers the existence of some error in my processes, for it is manifest that the general solution of the quintic (if such solution exists) must contain quintic radicals. But the error is not in the processes: these appear to be perfectly valid: the error is in the assumption that the sextic in t is an Abelian. That assumption is now proved to be false; and of course every conclusion which involves it must be fallacious.

In Part I., I have dealt with one of the trinomial form

$$x^5 - 5Qx^2 + E = 0,$$

to which the complete quintic can be reduced. That form enables us to deduce in a comparatively simple manner results which are perfectly general. Such results would no doubt be more valuable if calculated for the complete quintic, and Mr. Cayley's process of seminvariancy affords, as I have elsewhere‡ shown, substantial aid in such calculations. But there are many results which, with all the help of that process, it would be difficult if not impracticable to obtain for the perfect form.

I propose, therefore, again to make the above trinomial

* "Remarks on Mr. Harley's Paper on Quintics." By G. B. Jerrard, Esq.

† "Note on the Remarks of Mr. Jerrard." By James Cockle, Esq.

‡ In a Paper entitled "On the Method of Symmetric Products and on Certain Circular Functions connected with that Method," to be published shortly in the *Transactions of the Royal Society*.

quintic the basis of calculation. And to this end the notation of Part I. and the numbering of its articles are here resumed.

23. Since (Arts. 9, 10)

$$T = -2x^4 + 3^2 Qx,$$

we know that x can be expressed as a rational and integral function of T, and since T is the root of a quintic equation, we are permitted to assume

$$x = aT^4 + bT^3 + cT^2 + dT + e,$$

where a, b, c, d, e are functions of Q, E. In Art. 11, I have given two expressions for x which are rational but not integral with respect to T. Either of them might be rendered integral; for equating with

$$aT^4 + bT^3 + cT^2 + dT + e,$$

clearing of fractions, and reducing by means of the quintic in T, we are led to a quartic in T, the coefficients of which being equated to zero, furnish us with five simultaneous equations, linear in a, b, c, d, e, and from which the several values, in terms of Q, E, of these symbols might be obtained. But the same results may be arrived at more rapidly without the aid of the calculated value of x or the quintic in T, by the following method.

24. We have

$$x^5 = 5Qx^2 - E,$$

$$Tx = 3^2 Qx^2 - 2x^5,$$

$$= -Qx^2 + 2E,$$

$$x^5 = a(Tx)^4 + bx(Tx)^3 + cx^2(Tx)^2 + dx^3(Tx) + ex^4,$$

and thence we immediately deduce

$$\begin{aligned} 5Qx^2 - E = a\,&(Qx^2 - 2E)^4 \\ &- bx(Qx^2 - 2E)^3 \\ &+ cx^2(Qx^2 - 2E)^2 \\ &- dx^3(Qx^2 - 2E) \\ &+ ex^4. \end{aligned}$$

Developing, reducing by means of the trinomial equation, &c., we are conducted to

$$0 =$$

$$\begin{array}{r|r|r|r|r}
24\,Q^2E^2a & x^4 - 41\,Q^4Ea & x^3 + 25\,Q^6\ \ a & x^2 + 8\,Q^3E^2a & x - 5\,Q^5Ea \\
-\ 5\ \ \ Q^4b & -12\,QE^2b & -32\,QE^3a & +8\ \ \ E^3b & +16\ \ \ E^4a \\
-\ 4\,Q\,Ec & +\ 5\ \ \ Q^3c & +31\,Q^3Eb & -\ \ \ Q^2Ec & -\ 6\,Q^2E^2b \\
+\ \ \ \ \ \ e & +\ 2\ \ \ Ed & +\ 4\ \ \ E^2c & & +\ \ \ Q\,Ed \\
 & & -\ 5\ \ \ Q^2d & & +\ \ \ \ \ E \\
 & & -\ 5\ \ \ Q & &
\end{array}$$

an equation which must be satisfied identically. Equating, therefore, the coefficients of the several powers of x to zero, we find

$$a = \frac{2}{D}(Q^5 + 2^5E^3),$$

$$b = -\frac{2^5.5}{D}\ Q^3E^2,$$

$$c = \frac{2^4}{D}(Q^5 - 2^4.3E^3)\ QE,$$

$$d = \frac{1}{D}(Q^5 + 2^3.71E^3)\ Q^4,$$

$$e = -\frac{2^4}{D}(7^2Q^5 + 2^3.3^2E^3)\ Q^2E^2,$$

where, for shortness, D is written in place of

$$3^2Q^{10} - 2^7.23\,Q^5E^3 - 2^{10}E^6;$$

so that

$$\begin{aligned} x = \frac{1}{3^2Q^{10} - 2^7.23\,Q^5E^3 - 2^{10}E^6}\{2\,(Q^5 + 2^5E^3)\,T^4 & \\ -2^5.5\,Q^3E^2T^3 & \\ +2^4\,(Q^5 - 2^4.3E^3)\,QET^2 & \\ +(Q^5 + 2^3.71E^3)\,Q^4T & \\ -2^4\,(7^2Q^5 + 2^3.3^2E^3)\,Q^2E^2\}. & \end{aligned}$$

25. If we represent the roots of the sextic in t, viz.,

$$t_1,\ t_2',\ t_3',\ t_4,\ t_5,\ t_6',$$

or

$$t_1,\ -t_2,\ -t_3,\ t_4,\ t_5,\ -t_6,$$

by the numerals 1, 2, 3, 4, 5, 6 respectively, and write

$$-T = mn + pq + rs,$$

where, for the moment, m, n, p, q, r, s represent the six roots 1, 2, 3, 4, 5, 6 taken in an undetermined or arbitrary order of succession, then $-T$, or T, may be regarded as the root of a 15-ic equation (reducible to a quintic and a 10-ic), and its several values expressed as functions of the roots of the sextic in t, are

$$\left.\begin{aligned} -T_1 &= 13+25+46 \\ -T_2 &= 12+34+56 \\ -T_3 &= 14+26+35 \\ -T_4 &= 16+23+45 \\ -T_5 &= 15+24+36 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$\left.\begin{aligned} -T_6 &= 13+24+56 \qquad & -T_{11} &= 13+26+45 \\ -T_7 &= 12+35+46 \qquad & -T_{12} &= 12+36+45 \\ -T_8 &= 14+23+56 \qquad & -T_{13} &= 14+25+36 \\ -T_9 &= 16+25+34 \qquad & -T_{14} &= 16+24+35 \\ -T_{10} &= 15+26+34 \qquad & -T_{15} &= 15+23+46 \end{aligned}\right\} \ldots. (\beta).$$

The functions marked (α) are the several roots of the quintic in T, calculated in Art. 10. The several values of these functions in terms of the roots of the trinomial equation are given in Art. 9. The corresponding values of the ten functions marked (β) I now proceed to determine.

26. By Art. 9,

$$\tau_1\tau_2 + \tau_3\tau_4 + \tau_5\tau_6 = \mathrm{f}(25, 34, 1) + \mathrm{f}(12, 34, 5) + \mathrm{f}(15, 34, 2)$$
$$= \dot\Sigma x_1^2 x_2^2 + 3x_3^2 x_4^2 + x_1x_2x_5 \dot\Sigma x$$
$$+ x_3x_4 \dot\Sigma x^2 - 3x_3x_4 \dot\Sigma x_1x_2 - \dot\Sigma x^4,$$

where $\dot\Sigma$ is the usual symmetric symbol applied to three roots x_1, x_2, x_5 only. But, since the second, third, and fifth terms of the quintic are wanting, we have

$$\dot\Sigma x_1^2 x_2^2 = x_3^4 + x_4^4 + x_3^2 x_4^2, \qquad \dot\Sigma x^2 = -(x_3^2 + x_4^2),$$
$$x_1x_2x_5 \Sigma x = x_3^4 + x_4^4 + 2x_3x_4(x_3^2 + x_4^2) + 2x_3^2 x_4^2 - 5Q(x_3 + x_4),$$
$$\dot\Sigma x_1 x_2 = x_3^2 + x_4^2 + x_3x_4, \qquad \dot\Sigma x^4 = -(x_3^4 + x_4^4),$$
$$x_3^4 + x_4^4 = -x_3x_4(x_3^2 + x_4^2) - x_3^2 x_4^2 + 5Q(x_3 + x_4).$$

Hence, by substitution, we find

$$\tau_1\tau_3+\tau_2\tau_4+\tau_5\tau_6=5\{-x_3x_4(x_3^2+x_4^2)+2Q(x_3+x_4)\};$$

and, since the sinister member of this equation

$$=5(t_1t_3+t_2t_4+t_5t_6)=-5T_6,$$

therefore $\quad T_6=x_3x_4(x_3^2+x_4^2)-2Q(x_3+x_4).$

So that, writing for shortness

$$(mn)=x_mx_n(x_m^2+x_n^2)-2Q(x_m+x_n),$$

and deducing, by the theory of interchanges, the other values of T, we have

$$\begin{array}{ll} T_6=(34), & T_{11}=(25), \\ T_7=(45), & T_{12}=(13), \\ T_8=(15), & T_{13}=(24), \\ T_9=(35), & T_{14}=(12), \\ T_{10}=(14), & T_{15}=(23). \end{array}$$

27. It will be convenient now to calculate the equations for the sum and product (say s and p) of two roots of the quintic, and to express each of these quantities as a rational and integral function of the other.

Let the quintic be decomposed into a quadratic and a cubic; so that

$$x^5-5Qx^2+E=0=(x^2-sx+p)\left\{x^3+sx^2+(s^2-p)x+\frac{p(s^2-p)}{s}\right\}$$

$$=x^5-\frac{s^4-3s^2p+p^2}{s}x^2+\frac{p^2(s^2-p)}{s},$$

and therefore

$$s^4-3s^2p-5Qs+p^2=0\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$s^2p^2-Es-p^3=0\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then multiplying (1) into p, adding (2) to the product, and dividing the result by s, we find

$$s^3p-2sp^2-5Qp-E=0\ldots\ldots\ldots\ldots\ldots(3).$$

Next, combining (2) and (3) so as to express s as a rational function of p, we are conducted to

$$s=\frac{5Qp^4}{E^2-p^5}=\frac{E^2-p^5+5QEp}{5Qp^3},$$

and the equation in p is, therefore,

$$p^{10}-5^3Q^2p^7-5QEp^6-2E^2p^5+5QE^3p+E^4=0.$$

Moreover, expressing s as a rational and *integral* function of p, we have

$$s = \frac{p^2}{5QE^2}(-p^5 + 5^2Q^2p^2 + 5QEp + E^2).$$

Again, combining (1) and (2) so as to express p as a rational function of s, we find

$$p = \frac{2(s^3 - 5Q)s^2 - E}{5(s^3 + Q)} = \frac{\{(s^3 - 5Q)^2 - 3Es\}s}{2(s^3 - 5Q)s^2 - E};$$

consequently the equation in s is

$$s^{10} - 5Qs^7 - 11Es^5 - 5^2Q^2s^4 - 5.7QEs^2 + 5^3Q^3s - E^2 = 0;$$

and expressing p as a rational and integral function of s, we have

$$\begin{aligned} p = \frac{1}{5(E^3 + 2^6.3^3Q^5)}.\{&2^2.3QEs^9 + 2^4.3^2Q^3s^8 \\ &- E^2s^7 - 2^3.3^2Q^2Es^6 - 2^5.3^3Q^4s^5 \\ &- 2.3^2.7QE^2s^4 - 2^2.3.151Q^3Es^3 \\ &+ (2^4.3^2.5Q^5 + 13E^3)s^2 - 269Q^2E^2s \\ &- 2^6.3^3Q^4E\}. \end{aligned}$$

28. By means of the relations just established we can express the roots of the 10-ic in T as rational and integral functions of s or p only.

For, those roots (as we have shewn in Art. 26) are all included in the formula

$$(mn) = x_m x_n\{(x_m + x_n)^2 - 2x_m x_n\} - 2Q(x_m + x_n),$$

or, what is the same thing,

$$p(s^2 - 2p) - 2Qs,$$

which, since

$$-p^2 = s^4 - 3s^2p - 5Qs,$$

is equal to

$$2s^4 - 5s^2p - 12Qs;$$

and if in this expression we substitute for p, and reduce by means of the 10-ic in s, there results the formula

$$\begin{aligned} \frac{1}{E^3 + 2^6.3^3Q^5}\{&E^2s^9 + 2^2.3Q^2Es^8 + 2^4.3^2Q^4s^7 - 2.3QE^2s^6 \\ &- 2^3.3^2Q^3Es^5 - (11E^3 + 2^5.3^3Q^5)s^4 \\ &- 151Q^2E^2s^3 - 2^2.3.151Q^4Es^2 \\ &- 2^2.3Q(E^3 + 2.3.19Q^5)s - 2^4.3^2Q^3E^2\}, \end{aligned}$$

which includes all the roots of the 10-ic in T.

The corresponding formula in p is easily obtained; for, since

$$s^2p = \frac{Es}{p} + p^3,$$

therefore
$$T = \frac{E - 2Qp}{p}s - p^2$$

$$= \frac{p}{5QE^2}\{2Qp^6 - Ep^5 - 2.5^2Q^5p^3 + 3.5Q^2Ep^2 - 2QE^2p + E^3\}.$$

Employing Mr. Jerrard's $\mathfrak{S}$ process, or proceeding by elimination, either of the above expressions for T, combined with the equation in s or p, would enable us to calculate the 10-ic in T. The labor however threatens to be very great, even with all the help which Mr. Cayley's tables would afford. I pass on therefore to discuss the properties in θ and γ (α, β hereafter defined) corresponding to those in t and T just recorded.

29. In seeking to express the root of the trinomial equation as a rational and integral function of γ, it will be convenient first to find the corresponding expression in g' (Art. 18). In order to this let us write

$$x = ag'^4 + bg'^3 + cg'^2 + dg' + e,$$

where a, b, c, d, e are functions of the parameters Q, E. Then since

$$g' = -2Ex^3 + 3Q^2x^2,$$

we have, by substitution and reduction, as in Art. 24,

$$0 =$$

$$\left.\begin{array}{lr} & 2^4.3^3.13\,Q^6E^2a \\ - & 2^5.5\,QE^5a \\ - & 2.3^3.5\,Q^5Eb \\ + & 2^8E^4b \\ + & 3^2Q^4c \end{array}\right| x^4 \left.\begin{array}{lr} - & 3^3.7.29\,Q^8Ea \\ + & 2^4.5.37\,Q^3E^4a \\ + & 3^3.5\,Q^7b \\ - & 2^2.59\,Q^2E^3b \\ + & 2^2.5\,QE^2c \\ - & 2Ed \end{array}\right| x^3 \left.\begin{array}{lr} + & 3^4.5^2Q^{10}a \\ - & 2^4.3.5.59\,Q^5E^3a \\ + & 2^4E^6a \\ + & 2.3^2.53\,Q^4E^2b \\ - & 2^2.3.5\,Q^3Ec \\ + & 3Q^2d \end{array}\right| x^2$$

$$\left.\begin{array}{lr} + & 2^3.3^3.5\,Q^7E^2a \\ - & 2^4.31\,Q^2E^5a \\ - & 3^3Q^6Eb \\ + & 2^3.5\,QE^4b \\ - & 2^2E^3c \\ - & 1 \end{array}\right| x \begin{array}{lr} - & 3^4.5\,Q^9Ea \\ + & 2^3.3.109\,Q^4E^4a \\ - & 2^2.3^2.5\,Q^3E^3b \\ + & 2^23\,Q^2E^2c \\ + & e \end{array}$$

Whence, equating with zero the coefficients of the several powers of x in the dexter and resolving with respect to a, b, c, d, e, we find

$$a=\frac{3.5\,Q^3}{D}(3^5Q^{10}-2^3.3^2.5\,Q^5E^3+2^5E^6),$$

$$b=\frac{3\,Q^4E}{D}(3^7.17\,Q^{10}-2^4.3^3.5^3\,Q^5E^3+2^5.97\,E^6),$$

$$c=\frac{2E^2}{D}(3^7.5.7^2\,Q^{15}-2^2.3^4.13.181\,Q^{10}E^3+2^5.3.5.29\,Q^5E^6+2^7E^9),$$

$$d=-\frac{Q}{D}(3^9.5^3\,Q^{20}-2.3^6.4673\,Q^{15}E^3+2^4.3^3.5.17.181\,Q^{10}E^6$$
$$+2^6.3^2.191\,Q^5E^9-2^9.5E^{12}),$$

$$e=\frac{3\,Q^2E}{D}(3^9.5^2\,Q^{20}-2^2.3^6.5.103\,Q^{15}E^3$$
$$+2^6.3^3.667\,Q^{10}E^6+2^7.3^2.5^2\,Q^5E^9-2^{10}E^{12}),$$

where for shortness D is written in place of

$$(3^9.47\,Q^{20}-2^4.3^6.5.53\,Q^{15}E^3+2^8.3^3.371\,Q^{10}E^6$$
$$+2^9.3^2.5\,Q^5E^9-2^{10}E^{12})\,E^2.$$

80. Now, since

$$\gamma=5^4g=5^4(g'+2^3QE),$$

we have by an easy transformation,

$$x=\frac{1}{5^{16}}\{a\gamma^4-5^4(2^5QEa-b)\gamma^3$$
$$+5^8(2^7.3\,Q^2E^2a-2^3.3\,QEb+c)\gamma^2$$
$$-5^{12}(2^{11}Q^3E^3a-2^6.3\,Q^2E^2b+2^4QEc-d)\gamma$$
$$+5^{16}(2^{12}Q^4E^4a-2^9Q^3E^3b+2^6Q^2E^2c-2^3QEd+e)\}.$$

And if we replace a, b, &c. by their several values found in the last article, we get

$$x=\frac{1}{5^{15}E^2(3^9.47\,Q^{20}-2^4.3^6.5.53\,Q^{15}E^3+2^8.3^3.371\,Q^{10}E^6+2^9.3^2.5\,Q^5E^9-2^{10}E^{12})}$$
$$\times\{3\,Q^3(3^5Q^{10}-2^3.3^2.5\,Q^5E^3+2^5E^6)\gamma^4$$
$$-3.5^3Q^4E(3^3.7\,Q^{10}-2^4.3^2.5^2\,Q^5E^3+2^5.3^2.7\,E^6)\gamma^3$$
$$-3.5^7E^2(3^7.47\,Q^{15}-2^2.3^5.149\,Q^{10}E^3+2^5.3.59\,Q^5E^6-2^7E^9)\gamma^2$$

$$- 3.5^{11} Q (3^8.5^3 Q^{20} - 2.3^6.827\, Q^{15} E^3 + 2^4.3^2.1063\, Q^{10} E^6$$
$$- 2^6.3.433\, Q^5 E^9 + 2^9 E^{12}) \gamma$$
$$+ 5^{15} Q^2 E (3^9.5^2.42\, Q^{20} - 2^2.3^7.3727\, Q^{15} E^3$$
$$+ 2^6.3^3.2999\, Q^{10} E^6 - 2^7.3.139\, Q^5 E^9 - 2^{10}.7 E^{12})\},$$

an equation which corresponds to that in T given in Art. 24.

31. If we represent the roots of the sextic in θ by the numerals 1, 2, 3, 4, 5, 6, and write (see Art. 25)

$$\alpha_1 = 13 + 24 + 56, \qquad \beta_1 = 13 + 26 + 45,$$
$$\alpha_2 = 12 + 35 + 46, \qquad \beta_2 = 12 + 36 + 45,$$
$$\alpha_3 = 14 + 23 + 56, \qquad \beta_3 = 14 + 25 + 36,$$
$$\alpha_4 = 16 + 25 + 34, \qquad \beta_4 = 16 + 24 + 35,$$
$$\alpha_5 = 15 + 26 + 34, \qquad \beta_5 = 15 + 23 + 46,$$

we may regard α and β as the roots of a 10-ic, and the coefficients of this 10-ic may be calculated in terms of the parameters Q, E. We have already (Art. 15) expressed γ as a function of x. Following the method indicated in Art. 26, we are conducted to corresponding expressions for α and β. In effect, we have

$$\alpha_1 = 5^2 (\tau_1^2 \tau_3^2 + \tau_2^2 \tau_4^2 + \tau_5^2 \tau_6^2)$$
$$= 5^2 \{\mathrm{f}(25, 34, 1)\}^2 + 5^2 \{\mathrm{f}(12, 34, 5)\}^2 + 5^2 \{\mathrm{f}(15, 34, 2)\}^2$$
$$= 5^4 \{2QE - 3Qx_3^2 x_4^2 (x_3 + x_4) - Ex_3 x_4 (x_3 + x_4) - 12 Q^2 x_3 x_4\}.$$

So that if we write

$$\{mn\} = 5^4 \{2QE - 3Qx_m^2 x_n^2 (x_m + x_n) - Ex_m x_n (x_m + x_n) - 12 Q^2 x_m x_n\},$$

and deduce the other values of α and those of β by the doctrine of interchanges, we have

$$\alpha_1 = \{34\}, \qquad \beta_1 = \{25\},$$
$$\alpha_2 = \{45\}, \qquad \beta_2 = \{13\},$$
$$\alpha_3 = \{15\}, \qquad \beta_3 = \{24\},$$
$$\alpha_4 = \{35\}, \qquad \beta_4 = \{12\},$$
$$\alpha_5 = \{14\}, \qquad \beta_5 = \{23\}.$$

I may mention that these results are severally true for Mr. Cockle's α and β functions, the patterns given at the head of the article being derived from his by the θ inter-

changes indicated in the footnote under Art. 15. (See Mr. Cockle's "Observations," &c., *Philosophical Magazine*, Supplement, December, 1859.)

32. The roots (α, β) of the 10-ic are included in the formula

$$5^4(2QE - 3Qp^2s - Eps - 2^2.3Q^2p),$$

which since (Art. 27)

$$-p^2 = s^4 - 3ps^2 - 5Qs,$$

is equivalent to

$$5^4\{3Qs^5 - 3.5Q^2s^2 + 2QE$$
$$- (3^2Qs^3 + Es + 2^2.3Q^2)p\};$$

and if in this expression we substitute for p its value as a rational and integral function of s, and reduce by means of the equation in s, there results

$$\frac{5^3}{E^3 + 2^6.3^3Q^5}\{-2^2.3^2.5Q^3Es^9 + (E^3 - 2^4.3^3Q^5)s^8 + 3.5Q^2E^2s^7$$
$$+ 2^3.3^3.5Q^4Es^6 - 3^2Q(E^3 + 2^6.3^3Q^5)s^5 + 2.3^3.5.7Q^3E^2s^4$$
$$-E(13E^3 - 2^2.3^2.131Q^5)s^3 - 5Q^2(5.7E^3 - 2^4.3^3.11Q^5)s^2$$
$$+ 3.5.269Q^4E^2s + QE(7E^3 + 2^7.3^3.11Q^5)\}.$$

The equivalent expression in p is

$$\frac{5^3}{QE^2}(3Qp^9 + Ep^8 - 3.5^2Q^3p^6 - 2^3.5Q^2Ep^5$$
$$- 2^3QE^2p^4 - E^3p^3 - 2^2.3.5Q^3E^2p + 2.5Q^2E^3).$$

These formulæ correspond to those given in Art. 28 for the 10-ic in T.

33. Comparing T^2 with γ, we are conducted to some very interesting results. For, if we write T_{-1} in place of

$$\frac{1}{t_1t_3} + \frac{1}{t_2t_5} + \frac{1}{t^4t^6},$$

and refer to the formulæ for T_1 and γ_1 (Arts. 9, 15), we see at once that

$$x_1^2(3^2Q - 2x_1^3)^2 = T_1^2 = t_1^2t_3^2 + t_2^2t_5^2 + t_4^2t_6^2 + 2t_1t_2t_3t_4t_5t_6T_{-1}$$
$$= \frac{\gamma}{5^4} + 2.5Q^4T_{-1}$$
$$= -2Ex_1^3 + 3Q^2x_1^2 + 2^3QE + 2.5Q^4T_{-1};$$

and expanding the sinister member, we find, after making the necessary reductions,

$$T_{-1} = -\frac{1}{5Q^4}(Ex_1^2 + Q_2x_1^2 - 2^2QE).$$

In like manner we have

$$T_{-2} = \quad \frac{1}{t_1t_2} + \frac{1}{t_3t_4} + \frac{1}{t_5t_6} = -\frac{1}{5Q^4}(Ex_2^2 + Q_2x_2^2 - 2^2QE),$$

$$T_{-3} = -\frac{1}{t_1t_4} - \frac{1}{t_2t_6} + \frac{1}{t_3t_5} = -\frac{1}{5Q^4}(Ex_3^2 + Q^2x_3^2 - 2^2QE),$$

$$T_{-4} = \quad \frac{1}{t_1t_6} - \frac{1}{t_2t_3} - \frac{1}{t_4t_5} = -\frac{1}{5Q^4}(Ex_4^2 + Q^2x_4^2 - 2^2QE),$$

$$T_{-5} = -\frac{1}{t_1t_5} + \frac{1}{t_2t_4} - \frac{1}{t_3t_6} = -\frac{1}{5Q^4}(Ex_5^2 + Q_2x_5^2 - 2^2QE).$$

By the aid of the general formula in g, given in Art. 17, the equation in T_{-r} may be calculated without much labor; for if in that formula we make

$$a = 2^2,\ b = -1,\ c = -1,$$

there results a quintic whose root is

$$(g =)\ 2^2QE - Q^2x^2 - Ex^2,$$

and the passage from this equation to the quintic in T_{-r} is effected by simply writing $5Q^4T_{-r}$ for g. Making these substitutions, we find in fact,

$$\begin{aligned} 5^5Q^{20}T_{-r}^5 - 5^5Q^{17}ET_{-r}^4 + 5^4Q^{16}T_{-r}^3 & \\ + 5^2Q^5E(11E^5 - 2.19Q^5)\,T_{-r} & \\ - E^2(E^6 - 41Q^5E^3 - 19^2Q^{10}) & = 0. \end{aligned}$$

The function T_{-r} has ten other values corresponding to those of T_6, T_7, ... T_{15}. These values might be determined by a comparison of T_{-r} with α and β; and any one of them being expressed as a rational and integral function of the sum (s), or the product (p) only, of two roots of the trinomial, the 10-ic might be calculated by processes employed in preceding articles.

34. The results recorded in this paper, most of which were deduced shortly after the publication of Part. I., have been carefully verified by means of numerical examples; the particular equation

$$x^5 - \tfrac{5}{2}x^2 + \tfrac{3}{2} = 0,$$

of which two roots are unity and the others are given by the cubic

$$x^3 + 2x^2 + 3x + \tfrac{8}{5} = 0,$$

was found in general to afford a good test.

Mr. Cayley has calculated the reducing sextic in t, or rather that in $(\tau - \tau')^2$*, for the complete quintic

$$(a, b, c, d, e, f)(x, 1) = 0.$$

This sextic is found to take a remarkably simple form; it may in fact be regarded as a canonical in the theory of equations of the fifth degree. Mr. Cayley has also calculated the values, in terms of x, of the five combinations of the form

$$\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6,$$

where $\phi = \tau - \tau'$; and he has extended his researches on the subject to the corresponding properties of the quintic function

$$(a, b, c, d, e, f)(x, y)^5.$$

It now appears that for the determination of results for the complete quintic, the first of the two methods pointed out in the last article of Part I. is the more expeditious and effective; and that, in dealing with this form, Mr. Cayley's seminvariant process, alluded to in the introduction to the present paper, affords immense assistance.

Castle Hill House, Brighouse, Yorkshire,
July, 29, 1861.

ON THE DISCONTINUITY OF THE INTRINSIC EQUATIONS TO CURVES.

By WILLIAM WALTON, M.A., Trinity College.

IN Dr. Whewell's second Memoir† on the Intrinsic Equation to a Curve, he says:

"Scruples have been entertained by some of my readers as to whether we rightly suppose the portion of curve, added after reaching a cusp, to be negative. It has been said that

* It is proper to notice that

$$t = \frac{1}{2\sqrt{(5)}}(\tau - \tau') = \frac{1}{2\sqrt{(5)}}\Sigma x_1(x_2 - x_3).$$

† *Cambridge Philosophical Transactions*, Vol. IX., p. 150.

every cusp may be conceived to be the remnant left by a loop when the breadth of the loop vanishes; and as in a looped curve the increment of the length of the arc could nowhere become negative, it ought not to do so in the ultimate form of the looped curve."

The views which I am disposed to entertain on this subject may be best exhibited by the discussion of one or two suggestive instances.

I will commence with the consideration of the Cardioid, the polar equation to which is

$$r = a(1 - \cos\theta),$$

and of which the form is exhibited in fig. 21.

We have $\qquad ds\cos\psi = dr = -ad\cos\theta \ldots\ldots\ldots\ldots\ldots(1).$

Again, $\qquad \tan\psi = \dfrac{rd\theta}{dr} = \tan\dfrac{\theta}{2},$

whence $\qquad \psi = n\pi + \frac{1}{2}\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$

Also, by the geometry,

$$\phi = \theta + \psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (2) and (3), we have

$$\psi = \tfrac{1}{3}(\phi + 2n\pi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (1) and (2) we see that

$$\begin{aligned} ds\cos\psi &= -ad\cos(2\psi - 2n\pi) \\ &= -ad\cos 2\psi \\ &= \;\; 2a\sin 2\psi d\psi, \end{aligned}$$

whence $\qquad ds = 4a\sin\psi d\psi,$

$$\begin{aligned} s &= c - 4a\cos\psi \\ &= c - 4a\cos\frac{\phi + 2n\pi}{3}, \text{ by (4)}, \end{aligned}$$

c being an arbitrary constant.

When θ increases from $+0$ to $2\pi - 0$, $n\pi + \frac{1}{2}\theta$ increases from $n\pi + 0$ to $(n+1)\pi - 0$: but ψ cannot lie without the limits $+0$ and $\pi - 0$, as is evident from the geometry: hence, by the equation (2), it appears that $n = 0$: thus the Intrinsic Equation to the Cardioid is

$$s = c - 4a\cos\frac{\phi}{3},$$

while θ ranges from $+0$ to $2\pi-0$, and ϕ accordingly, in virtue of the equations (2) and (3), ranges from $+0$ to $3\pi-0$. Putting $s=0$ when $\phi=+0$, we see that $c=4a$: hence, within the limits $\phi=+0$ and $\phi=3\pi-0$, we have, for the Intrinsic Equation to the Cardioid,

$$s=4a\left(1-\cos\frac{\phi}{3}\right).$$

Next conceive θ to range from $2\pi+0$ to $4\pi-0$: then, by the equation (2), ψ ranges from $(n+1)\pi+0$ to $(n+2)\pi-0$: but, by the geometry, ψ is confined between the limits $+0$ and $\pi-0$; hence $n=-1$: consequently, while θ ranges from $2\pi+0$ to $4\pi-0$, and accordingly ϕ ranges from $2\pi+0$ to $5\pi-0$, the Intrinsic Equation to the curve is

$$s=c-4a\cos\frac{\phi-2\pi}{3}:$$

when $\theta=2\pi-0$ and $\phi=3\pi-0$, $s=8a$: this must also be the value of s when $\theta=2\pi+0$ and $\phi=2\pi+0$: hence $c=12a$, and therefore, within the limits $\phi=2\pi+0$ and $\phi=5\pi-0$, the Intrinsic Equation becomes

$$s=4a\left(3-\cos\frac{\phi-2\pi}{3}\right).$$

When θ ranges from $4\pi+0$ to $6\pi-0$, n is equal to -2, and ϕ ranges from $4\pi+0$ to $7\pi-0$, and the Intrinsic Equation is

$$s=4a\left(5-\cos\frac{\phi-4\pi}{3}\right).$$

Proceeding in the same way, we see that, when ϕ ranges continuously from $2(n-1)\pi+0$ to $(2n+1)\pi-0$, the Intrinsic Equation will be

$$s=4a\left\{2n-1-\frac{\cos\phi-2(n-1)\pi}{3}\right\}.$$

It is important to observe that, at the end of the first complete revolution of OP, $\phi=3\pi-0$; and that at the commencement of the second revolution of OP, $\phi=2\pi+0$. This shews that ϕ is diminished by π as P passes through the cusp. Owing to this discontinuity in the value of ϕ, the Intrinsic Equation to the curve, as might have been anticipated by the nature of the case, is discontinuous. In other words, there is no such a thing as "an Intrinsic Equation" to the curve: we ought in fact, with reference to the cusp, to speak of "the Intrinsic Equations" to the curve. In order

to render the geometrical conception of the loss of π, incurred by ϕ at the cusp, the more satisfactory, I have drawn in the figure a pectinated tangent to the curve, the arrow indicating the direction in which it is travelling round the curve. It will be observed that the tangency takes place on the side of the tangent opposite to the teeth of the comb. Conceive the pectinated tangent to arrive again at the cusp. Just before it reaches the cusp, the teeth of the comb point upwards; just after it passes it, the teeth point downwards. Attending to the direction of the arrow, it is evident that, in the passage of the pectinated tangent through the cusp, ϕ has been diminished in magnitude by the amount π. The instant ϕ has been thus diminished, the pectinated tangent has resumed the actual position which it occupied at the commencement of its progress from the cusp. The remarks which I have made on the subject of ϕ's sudden loss of π in the tangent's first transit through the cusp are of course equally applicable to all its subsequent transits.

From what has been said above it is manifest that the portion of s, generated after reaching the cusp, can in no sense be regarded as negative.

I will now proceed to the consideration of the Intrinsic Equation to the Cycloid.

The equations to the curve are

$$x = a(1 - \cos\theta), \quad y = a(\theta + \sin\theta).$$

Hence $$\tan\phi = \tan\frac{\pi - \theta}{2},$$

and therefore $$\theta + 2\phi = (2n+1)\pi \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The Intrinsic Equation is

$$s = c + 4a\cos\phi,$$

where c is a constant.

When P (fig. 22) is at E, $\theta = -\pi$: let π be the corresponding value of ϕ: then, by (1), $\pi = (2n+1)\pi$, and therefore $n = 0$. Moreover, there is no discontinuity in ϕ as P passes from E to F. Hence, throughout the arc EAF, $n = 0$; the value of ϕ, as P passes from E to F, ranging continuously from π to $+0$.

Next, let $n = 1$: then the equation (1) becomes

$$\theta + 2\phi = 3\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The relation $\theta + 2\phi = \pi$ belongs, as we have seen, to the arc EAF: the relation (2) therefore must correspond to a

different arc of the curve. Since, in considering the equation (2), we must reject the value $\pi - 0$ of θ, as belonging to the arc EAF, let us suppose $\theta = \pi + 0$, or that P has just passed through F: then, by (2), $2\phi = 2\pi - 0$, $\phi = \pi - 0$. Thus the transit of P through F changes ϕ from $+0$ to $\pi - 0$: that is, the pectinated tangent commences its movement through the arc FBG in the very altitude which it possessed at the beginning of its movement through EAF. Like remarks are applicable to all successive arcs: in the transit through each cusp, ϕ is augmented by π, in virtue of an opposite rotation of the tangent through two right angles. Thus the relation between θ and ϕ, which corresponds to the n^{th} arc, is

$$\theta + 2\phi = (2n - 1)\pi.$$

Thus the Intrinsic Equation for the expression of the whole length of the arc from A to any point in the n^{th} arc is

$$s = 4a\{2(n-1) + \cos\phi\},$$

the value of ϕ lying between π and 0.

The view which I have taken of the Intrinsic Equation to the cycloid seems to be more satisfactory than that which Dr. Whewell has adopted, and which supposes a change of sign in the successive arcs.

(1) I escape the objection to which Dr. Whewell's view is subject on the ground that a cusp is a degenerate successor of a loop.

(2) I give a geometrical interpretation to the multiple values of π in the equation $\theta + 2\phi = (2n + 1)\pi$.

(3) The different altitudes of the pectinated tangent are discriminated by my interpretation.

(4) In the analogous instance of the cusp in the cardioid, the discontinuity of ϕ in a transit through the cusp seems to be incontrovertible.

ON LINES OF CURVATURE AND GEODESIC LINES.

By MATTHEW I. JOYCE, B.A., Fellow of Gonville and Caius College.

IF in a plane curve an equilateral polygon be inscribed, we may suppose the curve to be made up of the middle points of the sides of the polygon taken consecutively, the sides produced to be tangents, and lines at right angles to them through their middle points to be normals; suppositions which, though not absolutely true, are so in the limit when the sides of the polygon are indefinitely diminished. The circle of curvature at any point will then be the limit of that equilateral and equiangular polygon which has two consecutive sides or elements common with the original curve, and whose centre will therefore coincide with the point of intersection of two consecutive normals as previously defined.

Similar suppositions may be adopted in the case of a curve of double curvature. Thus a normal plane will be a plane drawn through the middle point of an element perpendicular to it, a plane which passes through two consecutive tangents will be an osculating plane, and the point of intersection of three consecutive normal planes will be the centre of a sphere on which a curve can be drawn having three consecutive elements common with the original curve, *i.e.*, it will be the centre of spherical curvature. In the case of a curve drawn on the surface, the tangent plane corresponding to any particular element will be a plane passing through it and cutting the surface in a conic of which that element is a diameter; and a straight line, through the middle point of the element and centre of the conic, perpendicular to the plane of section, will be the corresponding normal.

Let the line of intersection of two consecutive tangent planes be called the *conjugate line.* Then, if the normals to the surface at two consecutive points of the curve be equally inclined to the osculating plane which contains the corresponding elements, it is clear from symmetry that these elements will be equally inclined to the conjugate line; and the acute angles which they make with it will be in the same or opposite directions, or, in other words, the angles

which the *same part* of the conjugate line makes with the two elements will be equal or supplementary, according as the normals lie on the same or different sides of the osculating plane. If the normals lie on the same side they will intersect, and the limit of the polygon will be the line of curvature; if on different sides, then, when the several elements are indefinitely diminished, the normal at any point will ultimately lie in the osculating plane, and the limit of the polygon will be a geodesic line. *The fundamental property*, then, *of those polygons which become in the limit lines of curvature is that the angles which any two consecutive elements make with the same part of the corresponding conjugate line are equal to each other; and in those polygons which ultimately become geodesic lines, the same angles are supplementary.*

Many propositions for the proofs of which it has hitherto been usual to have recourse to analysis will now follow from simple geometrical considerations.

If the tangents to a line of curvature be all parallel to a fixed plane, i.e., *if the curve be plane, the normals to the surface along it will make a constant angle with a fixed right line, and conversely.* For the several elements being all equal, and each perpendicular to the corresponding normal, every two consecutive normals intersecting each other will be equally inclined to the plane of the curve.

If the normals to the surface along a geodesic line be all parallel to a fixed plane, then the tangents to the curve will make a constant angle with a fixed right line, and conversely. For the normals to the surface being all parallel to a fixed plane, a normal to that plane will be parallel to all the tangent planes and conjugate lines, which will therefore be parallel to one another, so that each element will be equally inclined to the two conjugate lines which it meets: and since, by the fundamental property of a geodesic line, every two consecutive elements are equally inclined to the conjugate line which passes through their point of intersection, the several elements will make a constant angle with the normal to the fixed plane.

If the intersection of any two surfaces be a line of curvature on both, the surfaces will intersect throughout at a constant angle. For let LM, MN (fig. 23) be consecutive elements of the curve of intersection. With centre M and radius ML or MN describe a sphere and let it meet the conjugate lines for the two surfaces in P and Q respectively, join PL, PQ, PN, QL, and QN by arcs of great circles. Then since LMN is a line of curvature on either surface, the

arc PL is equal to the arc PN, and the arc QL to the arc QN; therefore, the arc PQ being common to the two spherical triangles PLQ and PNQ, the angle PLQ is equal to the angle PNQ; that is to say, if through any element of the curve of intersection of the two surfaces the tangent planes be drawn, the angle between them is constant.

The same is also true for a geodesic line, but in this case the surfaces will either touch each other, or intersect in a right line. For let LM, MN (fig. 24) be consecutive elements of the curve of intersection, MP the conjugate line for one surface; then the arcs PL, PN are supplementary, as are also the angles between ML, MN and the conjugate line for the second surface, MQ suppose. If now the lines ML, MP, and MN be not in one plane, *i.e.*, if LMN be not a right line, it is easily seen that the two conjugate lines must coincide and the surfaces touch each other, or else upon the same base, viz., the arc PQ, and upon the same side of it, there would be two spherical triangles having their sides terminated in one extremity of the base equal to one another and also those terminated in the other extremity. But if the elements LM, MN be in the same straight line, consecutivé tangent planes to either surface coincide, and the surfaces intersect throughout at a constant angle.

If two surfaces intersect throughout at a constant angle, and the curve of intersection be a line of curvature on the one, it will be a line of curvature on the other also. For if LMN (fig. 25) be a line of curvature on the one surface and therefore the arcs PL, PN equal, and the angles PLQ, PNQ be also equal, then from symmetry the arcs QL, QN will be equal, and LMN will be a line of curvature on the second surface also.

If two surfaces touch each other, and the curve of contact be a geodesic line on the one, it will also be a geodesic line on the other. For all the tangent planes and therefore the conjugate lines for the two surfaces coincide. The case in which two surfaces intersect in a right line requires no consideration.

It may not, perhaps, be unnecessary to remark that as we have always supposed it possible to draw a tangent *plane* to the surface through every element of the curve traced on it, the reasoning employed in some of the preceding propositions may not be applicable at a singular point.

When a curve is drawn on an ellipsoid, let the central radius vector parallel to the tangent at any point be denoted

by d, and let p be the perpendicular from the centre on the tangent plane to the surface: then

For all points of either a line of curvature or a geodesic line on an ellipsoid the product $p.d$ is constant. For let LM, MN (figs. 23 and 24) be consecutive elements of a curve traced on an ellipsoid, MP the corresponding conjugate line, then if LMN be a line of curvature or a geodesic line, the angles PML, PMN will be equal or supplementary. Let O (fig. 25) be the centre of the ellipsoid, and through m, the extremity of that central radius vector which is parallel to the conjugate line, draw a plane parallel to the osculating plane LMN, and therefore cutting the surface in a polygon similar and similarly situated to that in which it is cut by the osculating plane, because the sections of an ellipsoid made by parallel planes are similar and similarly situated ellipses; *i.e.*, the consecutive tangents ml, mn suppose, will be respectively parallel to ML and MN, and therefore the angles Oml, Omn will be equal or supplementary. In either case the perpendiculars from O on ml, mn, produced if necessary, will be equal, and therefore, if a plane be drawn through the centre parallel to the tangent plane at any point of the original curve, the perpendicular from the centre on that tangent to the central section which is parallel to the tangent to the original curve will be of constant length. Call this perpendicular p'. Then $p'pd$ is the volume of the enveloping parallelopiped which is constant. But p' has been shewn to be constant, therefore pd is constant.

If U, V be adjacent umbilici of an ellipsoid, P a point on the surface such that the sum of the geodesic arcs PU, PV is constant, then the locus of P is a line of curvature. The product pd is the same for all the tangents to the ellipsoid at an umbilicus, because at such a point the tangent plane is parallel to one of the central circular sections; and d, as also p, is the same for both umbilici, therefore pd, being constant along either geodesic arc, is at the point P the same for both, and therefore also d is the same for both; that is to say, at the point P the tangents to the two curves are parallel to equal diameters of the central section parallel to the tangent plane, and coincide with equal diameters of the indicating conic. Consequently, the bisector of the external angle between the two tangents, *i.e.*, the tangent to the locus of P, coincides with an axis of the indicating conic, and is therefore a tangent to one of the lines of curvature through P.

GEOMETRICAL NOTES.

By J. McDOWELL, B.A., F.R.A.S., Pembroke College.

THE main object of this paper is to give a simple *geometrical* proof of the following theorem:

"The circle through the middle points of the sides of a triangle touches the inscribed circle and the three escribed circles."

This theorem has not, I believe, been hitherto proved *geometrically*. An analytical proof of it and of some of the properties proved below will be found in "Nouvelles Annales de Mathématiques," t. 1, pp. 79–84 and 196–198.

It has already been proved geometrically that the circle through the middle points of the sides also passes through the feet of the perpendiculars from the angles to the opposite sides of the triangle, and through the middle points of the segments of these perpendiculars towards the vertices, that its centre is at the middle point of the line joining the centre of the circumscribed circle to the intersection of the perpendiculars, and its radius half that of the circumscribed circle.

These properties I shall assume. I shall also assume some other simple theorems which ought to be found in any elementary collection of geometrical principles.

Their proof, by simple geometry, can present no difficulty to the readers of this *Journal*.

Let R, r be the radii of the circumscribed and inscribed circles of the triangle ABC, O, O' their centres (figs. 26 and 27), and Pu the diameter perpendicular to BC.

Draw $O'H$, AE perpendicular to BC; AX perpendicular to Pu, and uV perpendicular to AB.

By known geometrical theorems, BV = half difference of sides AB, AC; AV = half their sum; and $BU = O'u = Cu$; $AO'.O'u = 2Rr$; and therefore $OO'^2 = R^2 - 2Rr$. Also the triangles AEL, $O'HL$, BuV, and AuX are easily seen to be similar, and angle $BuV = Aux = LO'H = LAE = \frac{1}{2}$ the difference of the angles B and C of the triangle ABC, $DH = BV$.

Hence $\qquad DH : DE :: BV : AX.$

But $BV : AX :: Bu : uA$ by the similar triangles BuV and AuX;

therefore $$DH : DE :: Bu : uA = O'u : uA$$
$$= LO' : O'A;$$

(since $$O'u : uL :: Au : O'u,$$

and therefore $$O'u : O'L :: Au : AO')$$
$$= LH : HE;$$

therefore $$DH.HE = DE.LH \text{ ....(1).}$$

In fig. 27 BK is perpendicular to AC, and meets the perpendicular AE in G.

N is middle point of AG, so that $AN = NG = OD$, and $DN = R$ is the diameter of circle through middle points of the sides of triangle ABC, M its centre, $OM = MG$.

Produce NE until $MQ = ND$, then it is easy to see that the triangles $LO'H$ and EDQ are similar; therefore

$$DE : EQ :: HO' (= r) : LH;$$

therefore $$DE.LH = r.EQ;$$

therefore by (1), $$r.EQ = DH.HE \text{ ....(2),}$$

add the rectangle $r.NE$ to these equals, then

$$rR = r.NE + DH.HE \text{ ....(3).}$$

Again $$O'N^2 + DH'^2 = DH'^2 + HE^2 + (NE - r)^2$$
$$= DE^2 - 2DH.HE + NE^2 - 2r.NE + r^2$$
$$= R^2 - 2DH.HE - 2r.NE + r^2$$
$$= R^2 - 2R.r + r^2 \text{ by (3)}$$
$$= (R - r)^2;$$

therefore $$O'N^2 + O'D^2 = (R - r)^2 + r^2,$$

but $$O'N^2 + O'D^2 = 2MO'^2 + 2MD^2 = 2MO'^2 + \tfrac{1}{2}R^2,$$

since $$DM = MN = \tfrac{1}{2}R;$$

therefore $$\tfrac{1}{2}R^2 + 2MO'^2 = R^2 - 2Rr + 2r^2,$$

and therefore $$O'M = \tfrac{1}{2}R - r \text{ .... (4),}$$

and therefore the circle on the diameter DN touches the inscribed circle *externally*, the inscribed circle being entirely within it.

Suppose now O_1 to be the centre of an escribed circle, and r_1 its radius; by changing the sign of r in (4), we have

$$O_1M = \tfrac{1}{2}R + r_1;$$

therefore the circle through middle points of sides and the inscribed circle touch one another externally. Hence the theorem is proved, but as this last principle, viz., the change of r into $-r_1$, is not recognized by Euclid, I shall proceed to give a legitimate geometrical proof that the circle with centre M and radius $\frac{1}{2}R$ also touches the three escribed circles.

First I may remark that only *one* circle can be described through the points D and E, touching the scribed incircle.

For suppose HE less than HD, produce DE *through* E to a point Y, such that $DY.YE = YH^2$; the tangents from Y to the inscribed circle give the points of contact of the required circle, with the inscribed, but one of these points is H, and the circle through D, E, and H is therefore infinite; thus only *one finite circle* can be described through the points D and E to touch the inscribed circle. This circle is therefore the one with centre M and radius $\frac{1}{2}R$.

In fig. 27 take $DS = DH$, then S is the point of contact of the circle escribed to BC, and L is clearly a centre of similitude of this escribed circle and the inscribed circle.

By a known geometrical property,

$$ED.DL = DH^2;$$

therefore taking away DL^2 from these equals, we have

$$DL.LE = SL.LH;$$

therefore also by another known geometrical property, the circle through D and touching the inscribed circle and the circle escribed to BC must also pass through E, but by what has been just proved, this is the circle with centre M and radius $\frac{1}{2}R$. Q.E.D.

Fig. 26. $AV^2 - BV^2 = Au^2 - Bu^2 = Au^2 - UO'^2$,

that is, $\frac{1}{4}(c+b)^2 - \frac{1}{4}(c-b)^2$, or $bc = 2AO'.O'u + AO'^2$

$$= 4Rr + r^2(s-a)^2 \ldots\ldots\ldots (5);$$

therefore $ab + ac + bc = 12Rr + 3r^2 + (s-a)^2 + (s-b)^2 + (s-c)^2$

$$= 12Rr + 3r^2 - s^2 + a^2 + b^2 + c^2,$$

but $$a^2 + b^2 + c^2 + 2(ab + ac + bc) = 4s^2;$$

therefore by adding these equals and dividing by 3, we have

$$ab + ac + bc = 4Rr + r^2 + s^2 \ldots\ldots\ldots\ldots\ldots (6).$$

Also, by a known theorem in geometry,

$$ab + ac + bc = 2R(p + p' + p''),$$

where p, p', p'' are the perpendiculars from the angles of the triangle ABC on the opposite sides; therefore

$$2R(p+p'+p''-2r)=r^2+s^2 \ldots\ldots\ldots\ldots (7),$$

hence $a^2+b^2+c^2=4s^2-2(ab+ac+bc)$

$$=2s^2-8Rr-2r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Fig. 27. $\left.\begin{aligned} AO'^2+O'G^2&=2O'N^2+2GN^2\\ &=2O'N^2+2OD^2,\end{aligned}\right\}$

and $\qquad 2O'N^2+2DH^2=2(R^2-2Rr+r^2);$

therefore $AO'^2+O'G^2=2R^2-4Rr+2r^2-2DH^2+2OD^2$

$$=4R^2-4Rr+2r^2-\tfrac{1}{2}a^2-\tfrac{1}{2}(b-c)^2;$$

therefore $O'G^2=4R^2-4Rr+r^2-\tfrac{1}{2}a^2-\tfrac{1}{2}(b-c)^2-(s-a)^2.$

Therefore by (5),

$$\begin{aligned} O'G^2&=4R^2+4Rr+3r^2-\tfrac{1}{2}a^2-\tfrac{1}{2}(c+b)^2+(s-a)^2\\ &=4R^2+4Rr+3r^2-\tfrac{1}{2}\{(b+c)^2-a^2\}+s(s-2a)\\ &=4R^2+4Rr+3r^2-s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).\end{aligned}$$

Also $OG^2=2O'O^2+2O'G^2-4O'M^2$

$$\begin{aligned} &=2R^2-4Rr+8R^2+8Rr+6r^2-2s^2-(R-2r)^2\\ &=9R^2+8Rr+2r^2-2s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).\end{aligned}$$

"Four circles are described, each self-conjugate with respect to one of the triangles formed by four straight lines in the same plane; prove that the four circles have a common chord." (Set by the Rev. N. M. Ferrers for the Math. Tripos, 16th January, 1862.)

Fig. 28. Let the four straight lines form the quadrilateral $ABCD$. Let x be the middle point of BD, y of AC, and z of EF; x, y, z are known to be in a straight line.

Draw AK perpendicular to ED, and DN to EA; let these perpendiculars meet in H.

The circle with centre H and $(\text{rad.})^2=KH.HA=DH.HN$ is obviously self-conjugate to the triangle ADE.

Similarly find G the centre of the circle self-conjugate to the triangle CDF.

Because DxN and DxM are isosceles triangles; therefore

$Hx^2-xN^2=DH.HN=(\text{rad.})^2$ of circle self-conj. to triangle ADE,

$Gx^2-xM^2=DG.GM=(\text{rad.})^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ CDF.

But $xN=xM=xB$ or Dx; therefore

$$Hx^2-Gx^2=(\text{rad.})^2 \text{ of circle } (H)-(\text{rad.})^2 \text{ of circle } (G);$$

therefore x is a point on the radical axis of the two circles (H) and (G). Similarly y can be shewn to be a point on their radical axis; therefore the straight line xyz is the radical axis of each of the six pairs of circles in the question, can be shewn to be the straight line xyz. Q. E. D.

N.B.—Since the pole and its polar with respect to a *real* circle must be on the same side of the centre, it is clear that when the triangle is obtuse-angled, its self-conjugate circle is real and finite; when the triangle is right-angled, the self-conjugate circle degenerates into a point, viz. the vertex of the right angle, and when the triangle is acute-angled, the self-conjugate circle is altogether imaginary.

The Perse Grammar School,
January 28, 1862.

DETERMINATION OF THE TRILINEAR EQUATION OF THE AXES OF A CONIC SECTION.

By P. J. Hensley, B.A., Fellow of Christ's College, Cambridge.

For the notation used in this paper I must refer to a previous paper on the "Determination of the foci of a Conic Section."

Let the determinants $\frac{dP}{da}, \frac{dP}{db}, \frac{dP}{dc}; \frac{d^2P}{da^2}, \frac{d^2P}{db^2}, \frac{d^2P}{dc^2}$ of equations (8) of that paper be denoted by U, V, W; u, v, w.

So that those equations may be written

$$P\alpha^2 - UH\alpha + \tfrac{1}{2}uH^2 = P\beta^2 - VH\beta + \tfrac{1}{2}vH^2 = P\gamma^2 - WH\gamma + \tfrac{1}{2}wH^2.$$

Let these be put in the form

$$\begin{aligned}&(P\alpha - \tfrac{1}{2}UH)^2 + \tfrac{1}{4}H^2(2Pu - U^2)\\ =\,&(P\beta - \tfrac{1}{2}VH)^2 + \tfrac{1}{4}H^2(2Pv - V^2)\\ =\,&(P\gamma - \tfrac{1}{2}WH)^2 + \tfrac{1}{4}H^2(2Pw - W^2).\end{aligned}$$

Or making

$$\frac{\alpha_0}{U} = \frac{\beta_0}{V} = \frac{\gamma_0}{W} = \frac{H}{2P} \;\ldots\ldots\ldots\ldots\ldots (14),$$

$$\begin{aligned}P(\alpha - \alpha_0)^2 - P\alpha_0^2 + \tfrac{1}{2}uH^2 &= P(\beta - \beta_0)^2 - P\beta_0^2 + \tfrac{1}{2}vH^2\\ &= P(\gamma - \gamma_0)^2 - P\gamma_0^2 + \tfrac{1}{2}wH^2.\end{aligned}$$

Taking the different combinations of these equations they represent the three different rectangular hyperbolas, whose asymptotes are parallel to the lines

$$\alpha = \pm \beta \quad \beta = \pm \gamma, \quad \gamma = \pm \alpha,$$

and which are concentric, having the point $(\alpha_0, \beta_0, \gamma_0)$ for their common centre.

The intersection of any two of these hyperbolas will give the foci; in considering two of these it is evident from symmetry, that if two branches of the two hyperbolas intersect in a point, their opposite branches will intersect in another point in the same straight line with the centre, and equidistant from it; and the conjugate hyperbolas will intersect in two points lying in the same straight line with the centre, and at the same distance from it as the former two; also the straight line joining these two points will be at right angles to the straight line joining the other two.

The two former points are the two real foci, and the points of intersection of the impossible branches corresponding to the two latter are the two imaginary foci.

Moreover it will be seen from the manner in which equations (8) were obtained, that the point $(\alpha_0, \beta_0, \gamma_0)$ is also the centre of the original conic.

An axis of the conic section may therefore be obtained by joining the point $(\alpha_0, \beta_0, \gamma_0)$ with a point of intersection of two of the above hyperbolas.

Let λ, μ, ν be directing ratios of such a line, ρ the distance between $(\alpha, \beta_0, \gamma_0)$, and any point (α, β, γ) on it; so that

$$\frac{\alpha - \alpha_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{\nu} = \rho \quad \ldots\ldots\ldots\ldots (15).$$

Therefore if (α, β, γ) be the point of intersection of two of the hyperbolas, ρ must satisfy the equations

$$P\lambda^2\rho^2 - P\alpha_0^2 + \tfrac{1}{2}uH^2 = P\mu^2\rho^2 - P\beta_0^2 + \tfrac{1}{2}vH^2 = P\nu^2\rho^2 - P\gamma_0^2 + \tfrac{1}{2}wH^2,$$

and eliminating ρ, we get as the condition connecting $\lambda : \mu : \nu$ in order that equations (15) may represent an axis

$$P(\beta_0^2 - \gamma_0^2)\lambda^2 + P(\gamma_0^2 - \alpha_0^2)\mu^2 + P(\alpha_0^2 - \beta_0^2)\nu^2$$
$$= \tfrac{1}{2}H^2(v - w)\lambda^2 + \tfrac{1}{2}H^2(w - u)\mu^2 + \tfrac{1}{2}H^2(u - v)\nu^2.$$

Eliminating λ, μ, ν from this by means of equations (15), we shall obtain the equation representing the axes.

If α_0, β_0, γ_0 be eliminated by equations (14), the equation obtained is

$$(V^2 - W^2)(2P\alpha - UH)^2 + (W^2 - U^2)(2P\beta - VH)^2 + (U^2 - V^2)(2P\gamma - WH)^2$$
$$= 2P(v - w)(2P\alpha - UH)^2 + 2P(w - u)(2P\beta - VH)^2 + 2P(u - v)(2P\gamma - WH)^2 \quad \ldots\ldots\ldots(16).$$

The equation representing the two axes.

In this equation the coefficient of H^2 in the left-hand member vanishes, and the whole equation may be divided by P; if this be done, and P put $=0$, we obtain (after rejecting the factor H, denoting the axis at an infinite distance) the equation giving the transverse axis of the conic section when it is a parabola.

$$(V^2 - W^2)(2U\alpha - uH) + (W^2 - U^2)(2V\beta - vH) + (U^2 - V^2)(2W\gamma - wH) = 0 \quad \ldots\ldots(17).$$

ON THE CONICS WHICH PASS THROUGH THE FOUR FOCI OF A GIVEN CONIC.

By A. Cayley.

THE foci of a conic are the points of intersection of the tangents through the circular points at infinity; the pair of tangents through each of the circular points at infinity is a conic through the four foci; and we have thus two conics $P=0$, $Q=0$ passing through the four foci; the equation of any other conic through the four foci is of course $P+\lambda Q=0$; and in particular if λ be suitably determined this equation gives the axes of the conic.

I was led to develope the solution, in seeking to obtain the elegant formulæ given in Mr. P. J. Hensley's paper "Determination of the foci of the conic section expressed by trilinear coordinates," *Journal*, t. V., pp. 177—183, (March, 1862).

I take the coordinates to be proportionate to the perpendicular distances of the point from the sides of the fundamental triangle, each distance divided by the perpendicular distance of the side from the opposite angle. This being so, the equation of the line infinity is

$$x + y + z = 0.$$

And, α, β, γ denoting the sides of the fundamental triangle, the equation of the circle circumscribed about the triangle is

$$\frac{\alpha^2}{x}+\frac{\beta^2}{y}+\frac{\gamma^2}{z}=0.$$

The foregoing two equations determine the circular points at infinity; and if (x_1, y_1, z_1) are the coordinates, there is no difficulty in obtaining the system of values

$$\begin{aligned} x_1:y_1:z_1 = & \quad -\alpha \quad :\beta(\cos C+i\sin C):\gamma(\cos B-i\sin B)\\ = & \,\alpha(\cos C-i\sin C): \quad -\beta \quad :\gamma(\cos A+i\sin A)\\ = & \,\alpha(\cos B+i\sin B):\beta(\cos A-i\sin A): \quad -\gamma, \end{aligned}$$

where as usual $i=\sqrt{(-1)}$, and where A, B, C denote the angles of the triangle, so that the cosines and sines of these angles denote given functions of α, β, γ. The coordinates (x_2, y_2, z_2) of the other circular point at infinity are of course obtained by merely writing $-i$ for i. We find also

$$\begin{aligned} & x_1x_2 : y_1y_2 : z_1z_2 : y_1z_2+y_2z_1 : z_1x_2+z_2x_1 : x_1y_2+x_2y_1\\ & =-\alpha^2:-\beta^2:-\gamma^2:\beta^2+\gamma^2-\alpha^2:\gamma^2+\alpha^2-\beta^2:\alpha^2+\beta^2-\gamma^2, \end{aligned}$$

which are the formula chiefly made use of in the sequel.

Suppose now that the equation of the conic is

$$U=(a, b, c, f, g, h)(x, y, z)^2=0.$$

Then putting for a moment

$$\begin{aligned} U_1 &= (a, b, c, f, g, h)(x_1, y_1, z_1)^2,\\ W_1 &= (a, b, c, f, g, h)(x, y, z)(x_1, y_1, z_1), \end{aligned}$$

and the like as regards U_2 and W_2; the tangents from the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively are

$$\begin{aligned} UU_1-W_1^2=0,\\ UU_2-W_2^2=0, \end{aligned}$$

which are respectively pairs of lines intersecting in the four foci. And it is moreover clear that the equation of the axes is

$$U_2W_1^2-U_1W_2^2=0.$$

The foregoing equations may be written

$$\begin{aligned} (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(yz_1-y_1z,\ zx_1-z_1x,\ xy_1-x_1y)^2=0,\\ (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(yz_2-y_2z,\ zx_2-z_2x,\ xy_2-x_2y)^2=0, \end{aligned}$$

where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are the inverse system of coefficients.

These may be written

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x_1, y_1, z_1)^2 = 0,$$
$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x_2, y_2, z_2)^2 = 0,$$

where a, b, c, f, g, h are quadric functions of x, y, z, viz.

$$\begin{aligned}
\mathrm{a} &= \mathfrak{B}z^2 + \mathfrak{C}y^2 - 2\mathfrak{F}yz,\\
\mathrm{b} &= \mathfrak{C}x^2 + \mathfrak{A}z^2 - 2\mathfrak{G}zx,\\
\mathrm{c} &= \mathfrak{A}y^2 + \mathfrak{B}x^2 - 2\mathfrak{H}xy,\\
\mathrm{f} &= -\mathfrak{A}yz - \mathfrak{F}x^2 + \mathfrak{G}xy + \mathfrak{H}xz,\\
\mathrm{g} &= -\mathfrak{B}zx - \mathfrak{G}y^2 + \mathfrak{H}yz + \mathfrak{F}yx,\\
\mathrm{h} &= -\mathfrak{C}xy - \mathfrak{H}z^2 + \mathfrak{F}zx + \mathfrak{G}zy,
\end{aligned}$$

and this being so, I combine the two equations as follows:

$$\begin{array}{llll}
x_2^2 & (\mathrm{a}, \ldots)(x_1, y_1, z_1)^2 & + x_1^2 & (\mathrm{a}, \ldots)(x_2, y_2, z_2)^2 = 0,\\
y_2^2 & \quad ,, & + y_1^2 & \quad ,, \qquad = 0,\\
z_2^2 & \quad ,, & + z_1^2 & \quad ,, \qquad = 0,\\
y_2z_2 & (\mathrm{a}, \ldots)(x_1, y_1, z_1)^2 & + y_1z_1 & (\mathrm{a}, \ldots)(x_2, y_2, z_2)^2 = 0,\\
z_2x_2 & \quad ,, & + z_1x_1 & \quad ,, \qquad = 0,\\
x_2y_2 & \quad ,, & + x_1y_1 & \quad ,, \qquad = 0,
\end{array}$$

any one of which is the equation of a conic passing through the four foci; the current coordinates being always (x, y, z).

The first of these equations is

$$\mathrm{a}(-2x_1^2x_2^2) + \mathrm{b}(x_2^2y_1^2 + x_1^2y_2^2) + \mathrm{c}(x_2^2z_1^2 + x_1^2z_2^2)$$
$$+2\mathrm{f}(x_2^2y_1z_1 + x_1^2y_2z_2) + 2\mathrm{g}(x_2^2z_1x_1 + x_1^2z_2x_2) + 2\mathrm{h}(x_2^2x_1y_1 + x_1^2x_2y_2) = 0,$$

where the quantities multiplied by a, b, &c. are all of them easily expressible in terms of x_1x_2, y_1y_2, z_1z_2, $y_1z_2 + y_2z_1$, $z_1x_2 + z_2x_1$, $x_1y_2 + x_2y_1$, which are respectively proportional to given functions of (α, β, γ); and replacing for a, b, &c. their values, the equation is

$$\begin{aligned}
&(\mathfrak{B}z^2 + \mathfrak{C}x^2 - 2\mathfrak{F}yz).2\alpha^4\\
&+ (\mathfrak{C}x^2 + \mathfrak{A}y^2 - 2\mathfrak{G}zx).(\alpha^2 + \beta^2 - \gamma^2)^2 - 2\alpha^2\beta^2.\\
&+ (\mathfrak{A}y^2 + \mathfrak{B}x^2 - 2\mathfrak{H}xy).(\gamma^2 + \alpha^2 - \beta^2)^2 - 2\gamma^2\alpha^2.\\
&+ 2(-\mathfrak{A}yz - \mathfrak{F}x^2 + \mathfrak{G}xy + \mathfrak{H}xz).\alpha^2(\beta^2 + \gamma^2) - (\beta^2 - \gamma^2)^2\\
&+ 2(-\mathfrak{B}zx - \mathfrak{G}y^2 + \mathfrak{H}yz + \mathfrak{F}yx). - \alpha^2(\gamma^2 + \alpha^2 - \beta^2)\\
&+ 2(-\mathfrak{C}xy - \mathfrak{H}z^2 + \mathfrak{F}zx + \mathfrak{G}zy). - \alpha^2(\alpha^2 + \beta^2 - \gamma^2) = 0.
\end{aligned}$$

Now putting for shortness

$$\square = \alpha^4 + \beta^4 + \gamma^4 - 2\beta^2\gamma^2 - 2\gamma^2\alpha^2 - 2\alpha^2\beta^2,$$

so that $-\square$ is equal to sixteen times the square of the area of the fundamental triangle, the coefficient of x^2 is

$$= \mathfrak{B}(\square + 2\alpha^2\gamma^2) + \mathfrak{C}(\square + 2\alpha^2\beta^2) - 2\mathfrak{F}\{-\square - \alpha^2(\beta^2+\gamma^2-\alpha^2)\},$$

which is

$$= (\mathfrak{B} + \mathfrak{C} + 2\mathfrak{F})\,\square + 2\alpha^2\{\mathfrak{B}\gamma^2 + \mathfrak{C}\beta^2 + \mathfrak{F}(\beta^2+\gamma^2-\alpha^2)\}.$$

And reducing in a similar manner the other coefficients, the equation is

$$\square\,\{(\mathfrak{B}+\mathfrak{C}+2\mathfrak{F})x^2 + \mathfrak{A}(y+z)^2 - 2(\mathfrak{H}+\mathfrak{G})x(y+z)\} + 2\alpha^2\Theta = 0,$$

where for shortness

$$\begin{aligned}
\Theta = \;& x^2.\,\mathfrak{B}\gamma^2 + \mathfrak{C}\beta^2 + \mathfrak{F}(\beta^2+\gamma^2-\alpha^2)\\
&+ y^2.\,\mathfrak{C}\alpha^2 + \mathfrak{A}\gamma^2 + \mathfrak{G}(\gamma^2+\alpha^2-\beta^2)\\
&+ z^2.\,\mathfrak{A}\beta^2 + \mathfrak{B}\alpha^2 + \mathfrak{H}(\alpha^2+\beta^2-\gamma^2)\\
&+ yz.\,-2\mathfrak{F}\alpha^2 + \mathfrak{A}(\beta^2+\gamma^2-\alpha^2) - \mathfrak{H}(\gamma^2+\alpha^2-\beta^2) - \mathfrak{G}(\alpha^2+\beta^2-\gamma^2)\\
&+ zx.\,-2\mathfrak{G}\beta^2 - \mathfrak{H}(\beta^2+\gamma^2-\alpha^2) + \mathfrak{B}(\gamma^2+\alpha^2-\beta^2) - \mathfrak{F}(\alpha^2+\beta^2-\gamma^2)\\
&+ xy.\,-2\mathfrak{H}\gamma^2 - \mathfrak{G}(\beta^2+\gamma^2-\alpha^2) - \mathfrak{F}(\gamma^2+\alpha^2-\beta^2) + \mathfrak{C}(\alpha^2+\beta^2-\gamma^2),
\end{aligned}$$

or, what is the same thing, the equation is

$$\begin{aligned}
\square\,\{&(\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + 2\mathfrak{F} + 2\mathfrak{G} + 2\mathfrak{H})\,x^2\\
&- 2(\mathfrak{A} + \mathfrak{H} + \mathfrak{G})\,x(x+y+z) + \mathfrak{A}(x+y+z)^2\} + 2\alpha^2\Theta = 0,
\end{aligned}$$

or putting for shortness

$$\begin{aligned}
\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + 2\mathfrak{F} + 2\mathfrak{G} + 2\mathfrak{H} &= \mathfrak{P},\\
\mathfrak{A} + \mathfrak{H} + \mathfrak{G} &= \mathfrak{L},\\
\mathfrak{H} + \mathfrak{B} + \mathfrak{F} &= \mathfrak{M},\\
\mathfrak{G} + \mathfrak{F} + \mathfrak{C} &= \mathfrak{N},
\end{aligned}$$

so that in fact

$$\mathfrak{P} = \mathfrak{L} + \mathfrak{M} + \mathfrak{N},$$

the equation is

$$\square\,\{\mathfrak{P}x^2 - 2\mathfrak{L}x(x+y+z) + \mathfrak{A}(x+y+z)^2\} + 2\alpha^2\Theta = 0.$$

The equation with y_2z_2, y_1z_1 is in a similar manner found to be

$$\begin{aligned}
\square\,\{&\mathfrak{F}x^2 - (\mathfrak{C} + \mathfrak{G})\,y^2 - (\mathfrak{B} + \mathfrak{H})\,z^2 + (\mathfrak{A} + 2\mathfrak{F} + \mathfrak{G} + \mathfrak{H})\,yz\\
&+ (-\mathfrak{B} - \mathfrak{H} + \mathfrak{F})\,zx + (-\mathfrak{C} - \mathfrak{G} + \mathfrak{F})\,xy\} - (\beta^2+\gamma^2-\alpha^2)\,\Theta = 0,
\end{aligned}$$

or, what is the same thing,

$$\square\,[(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+2\mathfrak{F}+2\mathfrak{G}+2\mathfrak{H})\,yz$$
$$-\{(\mathfrak{G}+\mathfrak{F}+\mathfrak{C})\,y+(\mathfrak{H}+\mathfrak{B}+\mathfrak{F})\,z\}\,(x+y+z)$$
$$+\mathfrak{F}\,(x+y+z)^2]-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0,$$

or, finally,

$$\square\,\{\mathfrak{P}yz-(\mathfrak{N}y+\mathfrak{M}z)(x+y+z)+\mathfrak{F}\,(x+y+z)^2\}-(\beta^2+\gamma^2-\alpha^2)\,\Theta=0.$$

Hence the entire system of equations is

$$\square\,\{\mathfrak{P}x^2\qquad -2\mathfrak{L}x\;(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}+2\alpha^2\Theta=0,$$
$$\square\,\{\mathfrak{P}y^2\qquad -2\mathfrak{M}y(x+y+z)+\mathfrak{B}\,(x+y+z)^2\}+2\beta^2\Theta=0,$$
$$\square\,\{\mathfrak{P}z^2\qquad -2\mathfrak{N}z\;(x+y+z)+\mathfrak{C}\,(x+y+z)^2\}+2\gamma^2\Theta=0,$$
$$\square\,\{\mathfrak{P}yz-(\mathfrak{N}y\;+\mathfrak{M}z)(x+y+z)+\mathfrak{F}\,(x+y+z)^2\}-(\beta^2+\gamma^2-\alpha^2)\Theta=0,$$
$$\square\,\{\mathfrak{P}zx-(\mathfrak{L}z\;+\mathfrak{N}z)\;(x+y+z)+\mathfrak{G}\,(x+y+z)^2\}-(\gamma^2+\alpha^2-\beta^2)\Theta=0,$$
$$\square\,\{\mathfrak{P}xy-(\mathfrak{M}x+\mathfrak{L}y)\;\;(x+y+z)+\mathfrak{H}\,(x+y+z)^2\}-(\alpha^2+\beta^2-\gamma^2)\Theta=0,$$

which are the equations of six conics, each of them passing through the four foci.

From the first three of these we have

$$\frac{1}{\alpha^2}\{\mathfrak{P}x^2-2\mathfrak{L}x\;\;(x+y+z)+\mathfrak{A}\,(x+y+z)^2\}$$
$$=\frac{1}{\beta^2}\{\mathfrak{P}y^2-2\mathfrak{M}y\,(x+y+z)+\mathfrak{B}\,(x+y+z)^2\}$$
$$=\frac{1}{\gamma^2}\{\mathfrak{P}z^2-2\mathfrak{N}x\;\;(x+y+z)+\mathfrak{C}\,(x+y+z)^2\},$$

which, allowing for the difference of notation, are Mr. Hensley's equations; it appears by his investigation that their geometrical signification is as follows: viz., if for shortness we denote the equations by

$$A=B=C,$$

then if we consider the tangents parallel to the x-side of the fundamental triangle, and the tangents parallel to the y-side of the fundamental triangle, the equation $A=B$ is the locus of a point such that the feet of the perpendiculars let fall from it on the four tangents lie in a circle. And similarly for the equations $A=C$, $B=C$.

If we multiply the six equations by 1, 1, 1, 2, 2, 2 and add, we obtain the identical equation $0=0$; if we multiply

them by a, b, c, $2f$, $2g$, $2h$ and add, then, after some easy reductions, we obtain for the equation of a new conic passing through the four foci

$$\square\{\mathfrak{P}U + K(x+y+z)^2\} + 2S\Theta = 0,$$

where

$$U = (a, b, c, f, g, h)(x, y, z)^2,$$

K is the discriminant $abc - af^2 - bg^2 - ch^2 + 2fgh$, and

$$S = a\alpha^2 + b\beta^2 + c\gamma^2 - f(\beta^2 + \gamma^2 - \alpha^2) - g(\gamma^2 + \alpha^2 - \beta^2) - h(\alpha^2 + \beta^2 - \gamma^2),$$

or, what is the same thing,

$$S = (f + a - h - g)\alpha^2 + (g - h + b - f)\beta^2 + (h - g - f + c)\gamma^2.$$

It would be interesting to ascertain the geometrical significations of the six conics and of the last mentioned new conic.

2, Stone Buildings, W.C.,
March 13*th*, 1862.

GEOMETRICAL INVESTIGATIONS.

By J. McDowell, B.A., F.R.A.S., Pembroke College, Cambridge.

1. If the quadrilateral $ABCD$ (fig. 29) be circumscribable by a circle and the parallelogram $DFBE$ be completed, prove that

$$AB.BF + CB.BE = BD^2.$$

Draw Fx and Ey so that the quadrilaterals $AFxD$ and $DyEC$ may be each circumscribable by a circle, then evidently Fx and Ey are parallel, and therefore $Bx = Dy$. Also

$$AB.BF = DB.Bx \text{ and } CB.BE = DB.By = DB.Dx;$$

therefore $AB.BF + CB.BE = DB.Bx + DB.Dx = DB^2$.

Q. E. D.

2. If $OFAE$ (fig. 30) be a parallelogram, prove that

$$BA.AF + CA.AE = AO^2 + BO.OC.$$

Describe a circle about the triangle ABC and produce AO to meet it in D. Join DB, DC, and draw Fx, Ey so that the quadrilaterals $BFxD$ and $CEyd$ may be circumscribable by circles.

Then Fx and Ey are parallel, and therefore $Ay = Ox$; therefore

$$BA.AF = DA.Ax \text{ and } CA.AE = DA.Ay = DA.Ox,$$

therefore

$$\begin{aligned} BA.AF + CA.AE = DA.Ax + DA.Ox &= DA.AO, \\ &= AO^2 + AO.OD, \\ &= AO^2 + BO.OC. \end{aligned}$$

Q. E. D.

3. If $OFAE$ (fig. 31) be a parallelogram, prove that

$$BA.AF + CA.AE = DA.AO = AO^2 +$$

rectangle under segments of chord through O, of circle circumscribing the triangle ABC.

Making the same construction as in (2) we have similarly

$$BA.AF + CA.AE = DA.AO.$$

Q. E. D.

N.B. Theorem (1) is evidently only a particular case of this one.

4. If three parallels to the sides of a triangle be drawn through any point in its plane and meeting each two sides of the triangle, the sum of the three rectangles under their segments is equal to the rectangle under the segments of a chord of the circumscribing circle drawn through the same point.

Let LM, PR, QS (fig. 32) be the three parallels to the sides of the triangle ABC, drawn through O.

By (3) $\quad BA.AS + CA.AR = AO^2 + AO.OG,$

and the first side of this equality

$$\begin{aligned} &= (PO.OR + LA.AS) + (QO.OS + MA.AR) \\ &= PO.OR + QO.OS + LO.OM + OA^2 \text{ by (2)}; \end{aligned}$$

therefore $\quad LO.OM + PO.OR + QO.OS = AO.OG.$

Q. E. D.

Cor. Hence, if the sum of the three rectangles be constant $(= K^2)$, and if δ denote the distance of O from the centre of the circumscribed circle, we have obviously

$$AO.OG = R^2 \sim \delta^2, \text{ that is } K^2 = R^2 \sim \delta^2;$$

therefore the locus of O is a circle concentric with the circumscribing circle and whose $(\text{rad})^2 = R^2 \sim \delta^2$.

N.B. If O be not *within* the triangle ABC (fig. 33), then it is proved similarly that

$$LO.OM - PO.OR - QO.OS = AO.OG,$$

but the enunciation in (4) embraces this case also, by considering the segments PO and QO negative, and this remark is applicable to similar cases throughout this paper.

5. If the quadrilateral $ABCD$ (fig. 34) be circumscribable by a circle, and GH and FE be drawn through any point O parallel to any two adjacent sides BC and BA; prove that $GO.OH + EO.OF =$ rectangle under segments of chord of circumscribing circle through O.

Draw MN and KL parallel to the other two sides AD and DC of the quadrilateral. Therefore

$$FO.OE = BG.MA = AB.BG - GB.BM,$$

$$GO.OH = BF.CK = CB.BF - FB.BK;$$

therefore

$$GO.OH + FO.OE = (AB.BG + CB.BF) - (GB.BM + FB.BK),$$

but $AB.BG + CB.BF = BO^2 +$ rectangle under segments of chord through O (by 3), since $GBFO$ is a parallelogram with an angle coinciding with an angle of triangle ABC, and $GB.BM + FB.BK = BO^2$ by (1), since $MBKO$ is clearly circumscribable by a circle, and $GBFO$ is a parallelogram.

Hence $GO.OH + FO.OE =$ rectangle under segments of chord through O. Q. E. D.

The following theorem, due to the Rev. H. Holditch, M.A., President of Gonville and Caius College, is now easily proved.

6. The locus of a point in the plane of a quadrilateral capable of being circumscribed by a circle, such that, if four parallels be drawn through it to each side of the quadrilateral and meeting the two sides respectively adjacent, the sum of the four rectangles under their segments shall be constant, $(= K^2)$ is a concentric circle with $(\text{radius})^2 = R^2 \sim \frac{1}{2}K^2$, where R is the radius of the circumscribing circle.

Using the figure in (5) we have, by (5), the sum of the rectangles, viz., $K^2 =$ *twice* rectangle under segments of

chord through $O = 2(R^2 \sim \delta^2)$, where $\delta =$ distance of O from centre of circumscribed circle;

therefore $$\delta^2 = R^2 \sim \tfrac{1}{2}K^2.$$

Q. E. D.

The Perse School,
May 8, 1862.

P.S. I subjoin Mr. Holditch's proof of theorem (6), with his permission:

If a straight line be drawn through a point parallel to a side of a quadrilateral which may be circumscribed by a circle and cutting the adjacent sides in E, F (fig. 35); then if $EO.OF+$ the three other similar quantities be constant, the locus of O is a circle.

Proof. Let $y = a_2x + b_2$ be the equation to side 2 from any origin.

X, Y the coordinates of O.

x_1, y_1 coordinates measured from O;

therefore $$x = X + x_1 \text{ and } y = Y + y_1,$$

$$y_1 + Y = a_2(X + x_1) + b_2 \text{ or if } B_2 = a_2X + b_2 - Y;$$

$y_1 = a_2x_1 + B_2$ is the equation to side 2 measured from O, but $y_1 = a_1x_1$ is equation to EF; therefore $x_1 = \dfrac{B_2}{a_1 - a_2}$ is coordinate of F, and so $\dfrac{B_4}{a_1 - a_4}$ is that of E, and as these are of opposite signs when O is within the quadrilateral,

$$OE.OF = \frac{B_2B_4.(1 + a_1^2)}{(a_1 - a_2).(a_4 - a_1)}$$

$$= \frac{(a_2X + b_2 - Y).(a_4X + b_4 - Y).(1 + a_1^2).(a_2 - a_3).(a_3 - a_4)}{(a_1 - a_2)(a_2 - a_3)(a_3 - a_4)(a_4 - a_1)};$$

and if K be the sum of the four products

$$(a_2X + b_2 - Y)(a_4X + b_4 - Y).(1 + a_1^2).(a_2 - a_3).(a_3 - a_4)$$
$$+ \text{ the three others} = (a_1 - a_2).(a_2 - a_3).(a_3 - a_4).(a_4 - a_1).K,$$

or, if $a_2 = \tan A_2$,

$$(X \sin A_2 + b_2 \cos A_2 - Y \cos A_2).(X \sin A_4 + b_4 \cos A_4 - Y \cos A_4)$$
$$\times \sin(A_2 - A_3).\sin(A_3 - A_4) + \text{three other terms}$$
$$= \sin(A_1 - A_2).\sin(A_2 - A_3).\sin(A_3 - A_4).\sin(A_4 - A_1).K,$$

where $$A_2 - A_1 + A_4 - A_3 = \pi.$$

Advancing the dashes by two steps, the sum of the first and third terms

$$= (X \sin A_2 + b_2 \cos A_2 - Y \cos A_2).(X \sin A_4 + b_4 \cos A_4 - Y \cos A_4)$$
$$\times \{\sin(A_2 - A_3).\sin(A_3 - A_4) + \sin(A_4 - A_1).\sin(A_1 - A_2)\},$$

but $\sin(A_4 - A_1) = \sin(A_2 - A_3)$, and $\sin(A_1 - A_2) = \sin(A_3 - A_4)$; therefore first and third terms

$$= (X \sin A_2 + b_2 \cos A_2 - Y \cos A_2).(X \sin A_4 + b_4 \cos A_4 - Y \cos A_4)$$
$$\times 2 \sin(A_2 - A_3).\sin(A_3 - A_4),$$

and advancing a step, the sum of the second and fourth terms

$$= (X \sin A_3 + b_3 \cos A_3 - Y \cos A_3)(X \sin A_1 + b_1 \cos A_1 - Y \cos A_1)$$
$$\times 2 \sin(A_3 - A_4).\sin(A_4 - A_1),$$

and adding and dividing by

$$2 \sin(A_3 - A_4).\sin(A_4 - A_1),$$

$$\left.\begin{array}{l}(X \sin A_2 + b_2 \cos A_2 - Y \cos A_2)(X \sin A_4 + b_4 \cos A_4 - Y \cos A_4)\\ +\, (X \sin A_3 + b_3 \cos A_3 - Y \cos A_3)(X \sin A_1 + b_1 \cos A_1 - Y \cos A_1)\end{array}\right\}$$
$$= \sin(A_1 - A_2).\sin(A_2 - A_3).\frac{K}{2}.$$

The coefficient of $XY = -\sin(A_2 + A_4) - \sin(A_1 + A_3) = 0.$

That of $X^2 = \sin A_2 \sin A_4 + \sin A_1 \sin A_3$

$$= \sin A_2.\sin(A_3 - A_1 - A_2) + \sin A_1 \sin A_3$$
$$= \tfrac{1}{2}\cos(A_1 - A_3) - \tfrac{1}{2}\cos(2A_2 - A_1 - A_3)$$
$$= -\sin(A_1 - A_2).\sin(A_2 - A_3),$$

and the coefficient of Y^2 is the same, and therefore the locus of O is a circle. The coefficient of

$$X = b_1 \cos A_1 \sin A_3 + b_2 \cos A_2 \sin A_4 + b_3 \cos A_3 \sin A_1$$
$$+ b_4 \cos A_4 \sin A_2,$$

coefficient of

$$Y = -b_1 \cos A_1 \cos A_3 - b_2 \cos A_2 \cos A_4 - b_3 \cos A_3 \cos A_1 - b_4 \cos A_4 \cos A_2,$$

and the constant on the left side

$$= b_2 b_4 \cos A_2 \cos A_4 + b_1 b_3 \cos A_1 \cos A_3.$$

If now the side 1 be made the axis of x, and the intersection of the first and fourth sides the origin, a_1, b_1, b_4, A_1 vanish, and the equation is

$$(X^2 + Y^2)\sin A_2 . \sin(A_2 - A_3) + b_2 \cos A_2 \sin A_4 . X$$

$$- (b_2 \cos A_2 \cos A_4 + b_3 \cos A_3)\, Y = -\sin A_2 . \sin(A_2 - A_3) . \frac{K}{2}.$$

Let $AC = b$, $AB = c$, (fig. 36) and angle $CAB = A = A_4 - \pi$, B and C being the angles of the triangle ABC, and the equation easily becomes

$$\left(X - \frac{c}{2}\right)^2 + \left(Y - \frac{c}{2\tan C}\right)^2 = \frac{c^2}{4\sin^2 C} - \frac{K}{2},$$

a circle whose centre is that of the circumscribing circle and its radius $= \sqrt{\left(R^2 - \frac{K}{2}\right)}$, and as all the terms of the quadrilateral disappear, the locus is the same for every quadrilateral inscribed in the circle.

H. H.

ON CERTAIN RELATIONS BETWEEN THE TANGENT PLANES AND RADII OF THE WAVE-SURFACE AND THE ELLIPSOID OF CONSTRUCTION.

By WILLIAM WALTON, M.A., Trinity College.

CONCEIVE two conjugate plane waves to be traversing a biaxal crystal: draw two tangent planes to the ellipsoid of construction, parallel respectively to their two planes of polarization. I proceed to establish the three following theorems:

(1) The velocities of the two tangent planes of the ellipsoid are equal respectively to the velocities of the two waves.

(2) The velocities of the two radii of contact of the ellipsoid are equal respectively to the velocities of the two rays.

(3) The radius of either point of contact is a semi-axis of the centric plane section of the ellipsoid, taken at right angles to the direction of the corresponding ray.

Let α, β, γ be the direction-cosines of either plane of polarization. Then, by a known proposition, the velocity of the corresponding wave is equal to

$$(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)^{\frac{1}{2}}.$$

But the distance of the centre of the ellipsoid of construction from a tangent plane of which α, β, γ are the direction-cosines is, by a known property of ellipsoids, also equal to

$$(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)^{\frac{1}{2}}.$$

The theorem (1) is therefore true.

Again, the equation to the tangent plane to the ellipsoid is

$$\alpha x + \beta y + \gamma z = (a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)^{\frac{1}{2}};$$

but its equation is also

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1,$$

where x', y', z', are the coordinates of the point of contact. Hence

$$(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)^{\frac{1}{2}} = \frac{a^2\alpha}{x'} = \frac{b^2\beta}{y'} = \frac{c^2\gamma}{z'}.$$

Hence, r' being the distance of the point of contact from the centre of the ellipsoid,

$$r'^2 = \frac{a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2}{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2}.$$

But, in an article in the *Quarterly Journal*, Vol. v., p. 127, entitled "On certain Analytical Relations between Conjugate Wave-Velocities, Ray-Velocities, and Planes of Polarization," I have shewn that, r being the velocity of the corresponding ray,

$$r^2 = \frac{a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2}{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2}.$$

Hence, in accordance with the theorem (2), r_1, r_2, denoting the velocities of the two rays, and r_1', r_2' of the two radii of contact,

$$r_1 = r_1' \text{ and } r_2 = r_2'.$$

Again, let λ, μ, ν be the direction-cosines of the radius of either point of contact; then, if

$$(a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2)^{\frac{1}{2}} = P,$$

we have

$$\frac{a^2\alpha}{\lambda} = \frac{b^2\beta}{\mu} = \frac{c^2\gamma}{\nu} = P \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Moreover (see Griffin's *Double Refraction*, p. 11), if l, m, n be the direction-cosines of the wave-velocity, we have, Q denoting a certain symmetrical expression,

$$\frac{l}{\alpha(v^2 - a^2)} = \frac{m}{\beta(v^2 - b^2)} = \frac{n}{\gamma(v^2 - c^2)} = Q \ldots\ldots(2).$$

Also, if L, M, N be the direction-cosines of the ray corresponding to the point of contact on the ellipsoid (see Griffin's *Double Refraction*, p. 18),

$$\frac{L}{l} \cdot \frac{v^2 - a^2}{r^2 - a^2} = \frac{M}{m} \cdot \frac{v^2 - b^2}{r^2 - b^2} = \frac{N}{n} \cdot \frac{v^2 - c^2}{r^2 - c^2} = \frac{v}{r} \ldots\ldots(3).$$

From the relations (1), (2), (3), we see that

$$L\lambda + M\mu + N\nu = \frac{Qv}{Pr}\{\alpha^2a^2(r^2 - a^2) + \beta^2b^2(r^2 - b^2) + \gamma^2c^2(r^2 - c^2)\},$$

and therefore, observing that

$$r^2 = \frac{a^4\alpha^2 + b^4\beta^2 + c^4\gamma^2}{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2},$$

it follows that

$$L\lambda + M\mu + N\nu = 0.$$

Thus we see that the ellipsoidal radius is at right angles to the corresponding ray; but, by the well-known relation between the Ellipsoid of Construction and the Wave Surface, we know that the length of the ray is equal to a semi-axis of the plane section of the ellipsoid made by a plane at right angles to the ray: hence the ellipsoidal radius must coincide with a semi-axis of this section.

From the expressions for the wave-velocity and corresponding ray-velocity in terms of α, β, γ, we may prove also the following theorem:

(3) If v, v', v'' be the wave-velocities in any three directions, such that the corresponding planes of polarization form a rectangular system, and r, r', r'' be the respective ray-velocities, then

$$(rv)^2 + (r'v')^2 + (r''v'')^2 = a^4 + b^4 + c^4.$$

In fact

$$(rv)^2 = a^4\alpha^2 \;\; + b^4\beta^2 \;\; + c^4\gamma^2,$$
$$(r'v')^2 = a^4\alpha'^2 + b^4\beta'^2 + c^4\gamma'^2,$$
$$(r''v'')^2 = a^4\alpha''^2 + b^4\beta''^2 + c^4\gamma''^2,$$

and therefore, by addition,

$$(rv)^2 + (r'v')^2 + (r''v'')^2 = a^4 + b^4 + c^4.$$

January, 1862.

COLLECTED WORKS OF GAUSS.

THE Academy of Sciences of Göttingen announce the publication of the Collected Works of Gauss, including the manuscripts left at his decease. The Works will appear in seven volumes, quarto, under the titles,—

I. Disquisitiones Arithmeticæ.
II. Hohëre Arithmetik.
III. Analysis.
IV. Geometrie und Methode der kleinsten Quadrate.
V. Mathematische Physik.
VI. Astronomie.
VII. Theoria Motus Corporum Cœlestium.

The *Theoria Motus* to appear when the copyright interest therein has expired. The contents of the several volumes are given in No. 1348 of the *Astronomische Nachrichten*.

The Works are to be published by subscription, which may be either for the whole or for the separate volumes; the subscription price to be hereafter fixed, but to be at about the rate of four thalers for a volume of from fifty to sixty sheets. Subscriptions may be addressed, post-free, "An das Secretariat der Königlichen Gesellschaft der Wissenschaften zu Göttingen." On completion, the Works will be sold in the ordinary course, at the rate of six or seven thalers per volume. The impression has begun, and will be carried on so that the first six volumes may at latest appear within five years.

ON THE KINEMATICAL SOLUTION OF CERTAIN PROBLEMS IN PLANE ASTRONOMY.

By WILLIAM WALTON, M.A., Trinity College.

IN the year 1858, a solution of the Problem of the Retardation of Sun Rise was communicated to me by my lamented friend, R. L. Ellis, of Trinity College. His solution was based upon the principle of the composition of axes of rotation. He remarked also that such a method of solving problems of small errors in plane astronomy, in virtue of the geometrical conceptions which it involves, would be generally preferable to the ordinary practice of applying the formulæ of spherical trigonometry to small triangles. In fact the principle of his method is in exact accordance with the practical working of Astronomical Problems by the use of the Globes.

The solutions of a few ordinary problems, here subjoined, essentially analogous to the Problem of Retardation, will enable astronomical students to comprehend the method, and to apply it for themselves in like cases.

1. Given a small error in the observed altitude of a known star, to find the corresponding error in the hour angle. Maddy's *Astronomy*, by Hymers, p. 145.

Let Z (fig. 37) be the zenith, P the pole of the equator, and Q the pole of the zenith distance of the star S.

Let Δh be the error of hour angle corresponding to the error Δz of zenith distance.

If we impress upon the celestial sphere an un-screw rotation Δh round the axis P and a screw rotation Δz round the axis Q, it is evident that the resultant zenith distance of S is the same as at first: hence these two rotations must be equivalent to a rotation about an axis R, R being somewhere in the great circle of which SZ is an arc. But, by a proposition in dynamics, R must lie in the great circle through P and Q. Hence R coincides with the intersection of SZ, QP, produced. Hence, by a well known theorem,

$$\Delta h . \sin PR = \Delta z . \sin QR.$$

But, since Q is the pole of SR, $\sin QR = 1$ and $\sin PR = \cos PQ$: hence

$$\Delta h = \frac{\Delta z}{\cos PQ} = \frac{\Delta z}{\sin a . \cos l},$$

a being the azimuth of the star, and l the latitude of the place.

2. To determine the error of latitude as consequent upon an error in observing the altitude of a known star.

Let ZS (fig. 38), produced, meet the horizon in T. Let M be the pole of the meridian, and Q that of ZT.

Let Δl and Δz be the corresponding errors of latitude and zenith distance.

If we impress upon the celestial sphere an un-screw rotation Δl about M and a screw rotation Δz about Q, the motion of Z is evidently at right angles to ZT: the resultant axis of rotation is therefore in ZT: but, by a proposition in dynamics, it is in QT. Hence T is the resultant axis, and, accordingly,

$$\Delta l . \sin MT = \Delta z . \sin QT,$$

or, a being the star's azimuth,

$$\Delta l . \cos a = \Delta z.$$

3. When the latitude and time are determined from two altitudes and the time between, to find the errors caused by given small errors in the observed altitudes. Maddy's *Astronomy*, by Hymers, p. 150.

By problem (1), if h be inaccurate,

$$\Delta z = \Delta h . \sin a \cos l;$$

and, by problem (2), if l be inaccurate,

$$\Delta z = \Delta l . \cos a.$$

Hence, by the addition of small errors, if h and l be both inaccurate,

$$\Delta z = \Delta l . \cos a + \Delta h . \sin a \cos l,$$

an equation on which the solution of the problem is based.

4. To find the effect of parallax on the declination of a known star. Maddy's *Astronomy*, p. 184.

Let Q (fig. 39) be the pole of SZ, and D that of SP. Let Δz and $\Delta\delta$ be corresponding errors of zenith distance and declination. Impress an un-screw rotation $\Delta\delta$ about D and a screw rotation Δz about Q. Since the motion of S is evidently perpendicular to ZS, the resultant rotation must take place about an axis I, where I is somewhere in the circle SZ. But I must lie somewhere in the circle QD. Hence I must coincide with the intersection of these two circles. Accordingly

$$\Delta z . \sin QI = \Delta\delta . \sin DI.$$

Since SD and SQ are both quadrants, S is the pole of QI, and therefore SI is a quadrant: hence

$$\sin DI = \sin DSI = \cos PSZ,$$

and, QI being a quadrant, $\sin QI = 1$: thus

$$\Delta z = \Delta\delta . \cos PSZ,$$

a formula on which the solution of the problem is based.

January 9, 1862.

ON CERTAIN PROPERTIES OF THE ROOTS OF ALGEBRAIC EQUATIONS.

By JAMES COCKLE.

1. IN the following investigation it is supposed that $m-1$ is a prime, say n, and that m is the product of two equal or unequal primes, say p and q. By way of preliminary illustration I shall refer to some former researches.

2. In the concluding portion of my "Notes on the Higher Algebra," published in the *Quarterly Journal*, for October, 1861, I have shewn that, if (x) and $\phi(x)$ be two roots of a general equation of the prime degree n, then each term of the series

$$(x),\ \phi(x),\ \phi^2(x),\ \ldots\ \phi^{n-1}(x)$$

expresses a root of the equation, and every root of the equation is comprised in the series. The function ϕ, which is not in general algebraic, is known whenever the solution of the equation of the $(n-1)^{\text{th}}$ degree is known. Its form is such that

$$\phi^s(x) = \phi^r(x)$$

whenever

$$s \equiv r \pmod{n}.$$

3. Taking, as an example, the cubic

$$x^3 - 3x + 2a = 0,$$

of which the roots are x_0, x_1, x_2, write

$$x_0 = \sin\left(\frac{\sin^{-1}a}{3}\right),$$

$$x_1 = \sin\left(\frac{2\pi + \sin^{-1}a}{3}\right),$$

$$x_2 = \sin\left(\frac{4\pi + \sin^{-1}a}{3}\right);$$

then, since

$$\sin^{-1}x_0 = \frac{\sin^{-1}a}{3},$$

we have

$$x_1 = \sin\left(\sin^{-1}x_0 + \frac{2\pi}{3}\right) = \phi x_0,$$

and, consequently, by the theorem

$$x_2 = \phi^2 x_0 = \sin\left(\sin^{-1} x_0 + \frac{4\pi}{3}\right).$$

In other words, x_1 being once known as a function of x_0, x_2 is given *a priori* as a function of x_0; and the set x_0, ϕx_0, $\phi^2 x_0$ replaces x_0, x_1, x_2.

4. Again, I have shown in the same paper that if two of the roots, (x) and $\phi(x)$, of an equation of prime degree, can be expressed by the uniform functions

$$\chi(I, J, \ldots L), \quad \chi(I_1, J_1, \ldots L_1),$$

then all its roots are comprised in the series

$$\chi(I, J, \ldots L), \quad \chi(I_1, J_1, \ldots L_1), \quad \ldots \chi(I_{n-1}, J_{n-1}, \ldots L_{n-1}).$$

Let V be a functional symbol, and let

$$I_1 = V_I(I, J, \ldots L), \quad J_1 = V_J(I, J, \ldots L), \quad \ldots L_1 = V_L(I, J, \ldots L),$$

then

$$I_2 = V_I(I_1, J_1, \ldots L_1), \quad J_2 = V_J(I_1, J_1, \ldots L_1), \quad \ldots L_2 = V_L(I_1, J_1, \ldots L_1),$$

$$I_3 = V_I(I_2, J_2, \ldots L_2), \text{ \&c.} = \text{\&c.}$$

5. To take a simple case, let

$$(x) = \chi(I, J) = \chi(I + J),$$

$$I_1 = V_I(I) = I, \quad J_1 = V_J(J) = J + j,$$

then we shall have

$$I = I_1 = I_2 = \ldots = I_{n-1},$$

$$J_1 = J + j, \quad J_2 = J + 2j, \quad \ldots J_{n-1} = J + (n-1)j;$$

and, if $J = 0$, the series of roots will be

$$\chi\{I, 0\}, \quad \chi\{I, j\}, \quad \chi\{I, 2j\}, \quad \ldots \chi\{I, (n-1)j\},$$

or $$\chi\{I\}, \quad \chi\{I + j\}, \quad \chi\{I + 2j\}, \quad \ldots \chi\{I + (n-1)j\},$$

where $$\chi\{I + sj\} = \chi\{I + rj\},$$

if $$s \equiv r \pmod{n}.$$

All the roots are contained in this series, and it follows that, if (x) and $\phi(x)$ are contained anywhere in it, then all the roots are contained in it. Let (x) and $\phi(x)$, respectively, be represented by

$$\chi\{I + rj\}, \quad \chi\{I + tj\},$$

t being greater than r, and each less than n. Put

$$I + rj = H, \quad (t - r)j = h.$$

Then the two roots in question will be represented by

$$\chi\{H\} \text{ and } \chi\{H + h\},$$

consequently each term of the series

$$\chi\{H\}, \ \chi(H + h\}, \ \chi\{H + 2h\}, \ \ldots \chi\{H + (n-1)\,h\}$$

will represent a different root of the given equation. This series is equivalent to the former one, for each term may be put under the form

$$\chi\{I + rj + s\,(t - r)j\},$$

and, for each value of s,

$$r + s\,(t - r)$$

will give a different residue to the prime modulus n.

6. The foregoing cubic illustrates this. Put

$$H = \frac{\sin^{-1} a}{3}, \quad h = \frac{2\pi}{3},$$

$$\chi\{H + h\} = \sin\{H + h\};$$

then, from

$$x_0 = \sin\left\{\frac{\sin^{-1} a}{3}\right\},$$

and

$$x_1 = \sin\left\{\frac{\sin^{-1} a}{3} + \frac{2\pi}{3}\right\},$$

we are able to infer that

$$x_2 = \sin\left\{\frac{\sin^{-1} a}{3} + 2.\frac{2\pi}{3}\right\},$$

and that

$$x_r = \sin\left\{\frac{\sin^{-1} a}{3} + s.\frac{2\pi}{3}\right\},$$

provided that

$$s \equiv r \ (\text{mod. } 3).$$

7. Let m denote the degree of the general algebraic equation, put under the usual form, $fx = 0$. Let ϕx (a function which, for reasons given in my concluding "Notes," &c., must not satisfy the relation

$$x - \phi x = 0)$$

satisfy

$$f\phi x = 0.$$

Then the infinite series of equations

$$f\phi^2 x = 0, \quad f\phi^3 x = 0, \quad f\phi^4 x = 0, \text{ \&c.}$$

will be satisfied.

For, the given equation being general and its roots symmetrically involved in its $m+1$ arbitrary coefficients, those roots are, analytically speaking, undistinguishable from each other. And the coexistence of

$$fx = 0, \quad f\phi x = 0$$

depends, not upon the particular root x which we insert in those expressions, but upon the forms of the functions which enter into them. So that, if we insert ϕx instead of x, the equations

$$f\phi x = 0, \quad f\phi^2 x = 0$$

coexist, and $\phi^2 x$ is a root of $fx = 0$.

Hence, again, we infer that

$$f\phi^2 x = 0, \quad f\phi^3 x = 0$$

coexist, and, continuing the process, we conclude that all the, in general transcendental, functions of the series

$$x, \quad \phi x, \quad \phi^2 x, \quad \phi^3 x, \quad \ldots \phi^r x \ldots$$

represent roots of $fx = 0$.

8. But, m being finite, the number of roots of $fx = 0$ is finite, and the series will recur after a certain number of terms. Suppose that the series ceases to give distinct values when, but not before, we reach the r^{th} term, or that

$$\phi^r x = x.$$

Then r of the roots of $fx = 0$ will be expressed by

$$x, \quad \phi x, \quad \phi^2 x, \quad \ldots \phi^{r-1} x,$$

no two terms of which series are, by supposition, equal in value.

If we insert the root x_0, then r of the roots may be written

$$x_0, \quad \phi x_0, \quad \phi^2 x_0, \quad \ldots \phi^{r-1} x_0,$$

and, if x_1 be a root not contained in the last line, the series

$$x_1, \quad \phi x_1, \quad \phi^2 x_1, \quad \ldots \phi^{r-1} x_1$$

will furnish r roots of $fx = 0$, otherwise the root x_0 would possess properties not possessed by the root x_1, a supposition which would be [illegible] with the hypothesis that $fx = 0$

is a general equation and its roots, consequently, undistinguishable from each other by any analytical properties.

9. None of the roots

$$x_0,\ \phi x_0,\ \phi^2 x_0,\ \dots \phi^{r-1} x_0$$

are identical with any one of the roots

$$x_1,\ \phi x_1,\ \phi^2 x_1,\ \dots \phi^{r-1} x_1.$$

For, suppose that we have such a relation as

$$\phi^u x_0 = \phi^v x_1,$$

then, applying ϕ^{-v} to either side, we have also

$$\phi^{u-v} x_0 = x_1,$$

which, taking the exponent to the modulus r, leads to a result of the form

$$x_1 = \phi^w x_0,\ (w < r),$$

a result inconsistent with the supposition that x_1 forms no part of the series

$$x_0,\ \phi x_0,\ \phi^2 x_0,\ \dots \phi^{r-1} x_0.$$

10. Hence if x_2 forms no part of (is not contained in) the series in x_0 and x_1, nor x_3 of those in x_0, x_1 or x_2, ... nor x_s of those in $x_1, x_2, \dots x_{s-1}$, we may express rs of the roots of $fx = 0$ as follows:

$$\begin{array}{llll} x_0, & \phi x_0, & \phi^2 x_0, & \dots \phi^{r-1} x_0, \\ x_1, & \phi x_1, & \phi^2 x_1, & \dots \phi^{r-1} x_1, \\ \vdots & \vdots & \vdots & \quad\ \vdots \\ x_s, & \phi x_s, & \phi^2 x_s, & \dots \phi^{r-1} x_s. \end{array}$$

11. Let s be chosen so that, if possible,

$$rs < m \text{ and } r(s+1) > m,$$

and let x_{s+1} be one of the t roots not comprised in the rows of Art. 10. Then each term of the series

$$x_{s+1},\ \phi x_{s+1},\ \phi^2 x_{s+1},\ \dots \phi^{r-1} x_{s+1}$$

will be a root of $fx = 0$, which equation will consequently have

$$r(s+1),$$

or more than m, roots, unless such relations as

$$\phi^u x_a = \phi^v x_a,\ \text{ or } \phi^u x_a = \phi^v x_b$$

subsist. But we cannot have

$$\phi^u x_a = \phi^v x_a,$$

for this would require that

$$u \equiv v \text{ (mod. } r),$$

a congruence which does not hold for any of the $s+1$ rows in which we suppose the roots to be distributed. Nor can we have

$$\phi^u x_a = \phi^v x_b,$$

for, supposing, as we are at liberty to do, b to be greater than a, we are led to

$$x_b = \phi^{u-v} x_a,$$

or, taking the exponent to the modulus r,

$$x_b = \phi^w x_a, \quad (w < r),$$

a relation inconsistent with the supposition that x_b forms no part of any of the preceding series.

12. Hence s cannot be so chosen as to render

$$rs < m \text{ and } r(s+1) > m,$$

in other words, m is a multiple of r, and, since $m = pq$, we see that one at least of the conditions

$$r = m, \quad r = p, \quad r = q$$

must be satisfied.

But which? or, how many of these conditions may be satisfied in dealing with the same equation? In order to obtain an answer to these questions form the reduced equation (in which x_0 may represent any one of the roots)

$$\frac{fx - fx_0}{x - x_0} = F(x, x_0) = 0.$$

This reduced equation is of the degree $m-1$ or n, and is a general equation of that degree; for, of the $m+1$ coefficients of the original equation $fx = 0$, only one is wanting in the reduced equation, the coefficients of which, consequently, involve m, or $n+1$, arbitrary quantities, as well as the root x_0. Moreover all the roots of $fx = 0$, excepting x_0, are roots of $F(x, x_0) = 0$, and, n being prime and the solution of the general equation of the n^{th} degree being supposed to be known, all the roots of $F(x, x_0) = 0$ may be expressed in terms of any one of them, say (x), by means of a symbol Φ, which fulfils the functions of the symbol ϕ of my concluding

"Notes," &c., a symbol here appropriated to a somewhat different purpose. Let the series

$$(x),\ \Phi(x),\ \Phi^2(x),\ \dots \Phi^{n-1}(x)$$

represent the n roots of $F(x, x_0)=0$, then, n being prime, no two terms of this series are equal to each other. Hence the series

$$x_0,\ (x),\ \Phi(x),\ \Phi^2(x),\ \dots \Phi^{n-1}(x)$$

will represent the $n+1$ or m roots of $fx=0$. But ϕx_0 is a root of the latter equation, and, since (x) may be taken as any one of the n roots of $F(x, x_0)=0$ at pleasure, we may replace the series last above written by

$$x_0,\ \phi x_0,\ \Phi\phi x_0,\ \Phi^2\phi x_0,\ \dots \Phi^{n-1}\phi x_0,$$

which, in its turn, may be replaced by

$$x_0,\ \phi x_0,\ \phi_2 x_0,\ \phi_3 x_0,\ \dots \phi_n x_0.$$

13. In this series suffixes have replaced functional exponents because the mode in which $\phi_1, \phi_2, \dots \phi_n$ are obtained, leaves us without other information than that each of the series of the form

$$x_0,\ \phi_a x_0,\ \phi_a^2 x_0,\ \dots \phi_a^n x_0,$$

in other words, every one of the series

$$\begin{array}{llll}
x_0, & \phi x_0, & \phi^2 x_0, & \dots \phi^{m-1} x_0, \\
x_0, & \phi_2 x_0, & \phi_2^2 x_0, & \dots \phi_2^{m-1} x_0, \\
x_0, & \phi_3 x_0, & \phi_3^2 x_0, & \dots \phi_3^{m-1} x_0, \\
\vdots & \vdots & \vdots & \vdots \\
x_0, & \phi_{m-1} x_0, & \phi_{m-1}^2 x_0, & \dots \phi_{m-1}^{m-1} x_0
\end{array}$$

will consist entirely of roots of $fx=0$.

14. I proceed to show that, of these series, only $q-1$ will recur after the q^{th} term and only $p-1$ after the p^{th} term. Suppose that the roots of $fx=0$ are capable of being distributed into $\frac{m}{r}$, say $s+1$, groups of r roots in the two following ways:

$$\begin{array}{llll}
x_0, & \phi_a x_0, & \phi_a^2 x_0, & \dots \phi_a^{r-1} x_0, \\
x_1, & \phi_a x_1, & \phi_a^2 x_1, & \dots \phi_a^{r-1} x_1, \\
\vdots & \vdots & \vdots & \vdots \\
x_s, & \phi_a x_s, & \phi_a^2 x_s, & \dots \phi_a^{r-1} x_s,
\end{array}$$

and

$$\begin{array}{cccc} x_0, & \phi_b x_0, & \phi_b^2 x_0, & \dots \phi_b^{r-1} x_0, \\ x_e, & \phi_b x_e, & \phi_b^2 x_e, & \dots \phi_b^{r-1} x_e, \\ \vdots & \vdots & \vdots & \vdots \\ x_l, & \phi_b x_l, & \phi_b^2 x_l, & \dots \phi_b^{r-1} x_l; \end{array}$$

then no equations such as

$$\phi_a^u x_0 = \phi_b^v x_k, \quad \phi_a^u x_k = \phi_b^v x_0,$$

in which k is different from 0, can subsist, for they would in fact lead to relations between the two independent and arbitrary quantities x_0 and x_k. Yet each of these sets comprises all the roots of $fx = 0$. Hence such relations as

$$\phi_a^a x_0 = \phi_b^\beta x_0$$

must subsist, and that, too, in virtue of the forms of the functions ϕ_a and ϕ_b, and not of any particular value of x.

15. To either side of this last relation apply the equivalent operations $\phi_a^{(c-1)\alpha}$ and $\phi_b^{(c-1)\beta}$, and there results

$$\phi_a^{c\alpha} x_0 = \phi_b^{c\beta} x_0.$$

Now r being a prime, and α and β each less than r, we may always determine c so as to satisfy the congruence

$$c\beta \equiv 1 \text{ (mod. } r),$$

for $\beta, 2\beta, 3\beta, \dots (r-1)\beta$ each give a different residue with respect to r. And, c being so determined, we have

$$\phi_a^{c\alpha} x_0 = \phi_b x_0,$$

or

$$\phi_b x_0 = \phi_a^\gamma x_0,$$

where ($c\alpha$ being taken to the modulus r) γ is one of the numbers

$$1, 2, 3, \dots (r-1).$$

16. It follows that the series

$$x_0, \ \phi_a x_0, \ \phi_a^2 x_0, \ \dots \phi_a^{r-1} x_0$$

does not differ essentially from

$$x_0, \ \phi_b x_0, \ \phi_b^2 x_0, \ \dots \phi_b^{r-1} x_0,$$

for, from what has just been proved, it appears that the latter may be written

$$x_0, \ \phi_a^\gamma x_0, \ \phi_a^{2\gamma} x_0, \ \dots \phi^{(r-1)\gamma} x_0,$$

which, since

$$\gamma,\ 2\gamma,\ 3\gamma,\ \ldots (r-1)\gamma$$

give, in some undetermined order, the residues

$$1,\ 2,\ 3,\ \ldots (r-1)$$

with respect to 1, is substantially identical with the former series. It has however $r-1$ forms or arrangements, arising from the $r-1$ values of γ.

17. Hence, since we may replace r by p or q indifferently, only $q-1$ of the series of Art. 13 will recur after the q^{th} term, and only $r-1$ after the r^{th} term; the remaining

$$m-1-(p+q-2),\ \text{or}\ m-(p+q)+1,\ \text{or}\ pq-(p+q)+1$$

will not recur until the m^{th} term, *i.e.* the end of the series, is reached. And when $q=p$ only one of the series will recur after the p^{th} term. In other words, when p and q are unequal, the roots of $fx=0$ can only be resolved in one way into p rows of q roots, and in one way into q rows of p roots; and when $p=q$ there is only one way in which they can be resolved into p rows of p roots. Different values of γ give a different arrangement only, not different systems of functions. And all the modes in which the roots can be exhibited in a series of m terms are substantially the same, the order only of the roots being different: and each of the functions which occur in the other two series will be found in the m series.

18. We may illustrate this by the sextic

$$x^6-a^6=0.$$

Here $$m=6,\ p=3,\ q=2,$$

and if we denote by ω an unreal cube root of unity, the roots may be written

$$a,\ -a;\ \omega a,\ -\omega a;\ \omega^2 a,\ -\omega^2 a:$$

$$a,\ \omega a,\ \omega^2 a;\ -a,\ -\omega a,\ -\omega^2 a:$$

$$a,\ -\omega a,\ \omega^2 a,\ -a,\ \omega a,\ -\omega^2 a:$$

in the first of which rows we have a periodic function of the second order $\{$for $-(-a)=a\}$, in the second a periodic function of the third order, and in the last one of the sixth order. In general suppose that all the roots of a sextic are given by the series

$$x_0,\ \phi x_0,\ \phi^2 x_0,\ \phi^3 x_0,\ \phi^4 x_0,\ \phi^5 x_0,$$

then we may arrange them in either of the forms

$$x_0,\ \phi^3x_0;\ \phi x_0,\ \phi^3.\phi x_0;\ \phi^2x_0,\ \phi^3.\phi^2x_0:$$

$$x_0,\ \phi^2x_0,\ \phi^4x_0;\ \phi x_0,\ \phi^2.\phi x_0,\ \phi^4.\phi x_0:$$

i.e. in periods of either two or three.

19. Again, take the biquadratic

$$x^4 - a^4 = 0.$$

Here $$m = 4,\ p = q = 2,$$

and the roots are

$$a,\ -a;\ \sqrt{(-1)}.a,\ -\sqrt{(-1)}.a:$$

in which row the only possible periods of two are exhibited. The period of four is

$$a,\ \pm\sqrt{(-1)}.a,\ -a,\ \mp\sqrt{(-1)}.a.$$

In general let the roots of a biquadratic be

$$x_0,\ \phi x_0,\ \phi^2x_0,\ \phi^3x_0,$$

then we may arrange them in the form

$$x_0,\ \phi^2x_0;\ \phi x_0,\ \phi^2.\phi x_0:$$

which contains periods of the second order.

20. Passing down the rows of series given in Art. 13 until we find one of which the characteristic, say ϕ_r, is of the m^{th} order of periodicity, write ϕ instead of ϕ_r. And suppose that two of the roots of $fx = 0$, say $\phi^a x_0$ and $\phi^b x_0$, are expressed by the uniform functions

$$\chi\{I_a, J_a, \ldots L_a\},\quad \chi\{I_b, J_b, \ldots L_b\},$$

which are such that, V being a functional symbol,

$$\begin{aligned} I_b &= V_I(I_a, J_a, \ldots L_a),\\ J_b &= V_J(I_a, J_a, \ldots L_a),\\ &\ \vdots \qquad \vdots \qquad \vdots\\ L_b &= V_L(I_a, J_a, \ldots L_a).\end{aligned}$$

Then from

$$\phi^a x_0 = \chi\{I_a, J_a, \ldots L_a\}$$

we pass to

$$\phi^{b-a}.\phi^a x_0 = \phi^{b-a}.\chi\{I_a, J_a, \ldots L_a\},$$

or to $$\phi^b x_0 = \phi^{b-a}.\chi\{I_a, J_a, \ldots L_a\},$$

and $$\phi^{b-a}.\chi\{I_a, J_a, \ldots L_a\} = \chi\{I_b, J_b, \ldots L_b\}.$$

Hence $(\phi^{b-a})^2.\chi\{I_a, J_a, \dots L_a\} = \phi^{b-a}.\chi\{I_b, J_b, \dots L_b\}$

$$= \chi\{I_c, J_c, \dots L_c\},$$

where

$$\begin{aligned} I_c &= V_I\{I_b, J_b, \dots L_b\}, \\ J_c &= V_J\{I_b, J_b, \dots L_b\}, \\ &\vdots \\ L_c &= V_L\{I_b, J_b, \dots L_b\}; \end{aligned}$$

and it is to be observed that in these formulæ, as indeed in those of Arts. 4 and 5, the functions V_I, V_J, ... V_L may be of totally different kinds and a change in the suffixed letter may denote a change in the form of the function.

21. Continuing this process we shall be led to a set of uniform functions

$$\chi\{I_a, J_a, \dots L_a\},\ \chi\{I_b, J_b, \dots L_b\},\ \dots \chi\{I_l, J_l, \dots L_l\},$$

as the expressions for q roots, or p roots, or for all the m roots, of $fx = 0$; I_l, &c. being derived from the preceding I, &c. in the same way in which I_b, I_c, &c. were derived from I_a, I_b, &c. and the m^{th} operation at latest reproducing I_a, &c. inasmuch as

$$(\phi^{b-a})^m x_0 = \phi^{(b-a)m} x_0 = x_0.$$

22. If $b - a$ be equal to p or to q, we have

$$\phi^{(b-a)q} x_0 = x_0, \text{ or } \phi^{(b-a)p} x_0 = x_0;$$

but in other cases (*i.e.* when $b - a$ is prime to m) we can find a number g such that

$$(b-a)\,g \equiv 1 \ (\text{mod. } m),$$

and, by a modification of the present argument analogous to that made, in Art. 5, of a former one, we may conclude that to the series

$$x_0,\ \phi x_0,\ \phi^2 x_0,\ \dots \phi^{m-1} x_0$$

there corresponds a set of uniform functions, say

$$\chi_0, \chi_1, \chi_2, \cdots \chi_{m-1},$$

each derived from the one preceding by the same operation.

23. Suppose that

$$\phi^{b-a} = \phi^g = \psi, \text{ or } \phi^g x_0 = \psi x_0,$$

then, as we know from the foregoing discussion, all the m roots of $fx=0$ may be considered as contained in the q rows

$$\begin{array}{llll} x_0, & \psi x_0, & \psi^2 x_0, & \ldots \psi^{p-1} x_0, \\ x_1, & \psi x_1, & \psi^2 x_1, & \ldots \psi^{p-1} x_1, \\ \vdots & \vdots & \vdots & \quad \vdots \\ x_q, & \psi x_q, & \psi^2 x_q, & \ldots \psi^{p-1} x_q. \end{array}$$

We know, moreover, that if the series

$$\chi_0,\ \chi_1,\ \chi_2,\ \ldots$$

recurs after the p^{th} term, its p terms must coincide with the roots contained in some of the preceding rows. For, if we had, by way of example,

$$\chi_0 = x_0 \text{ and } \chi_1 = \psi x_1,$$

then, remembering that x_1 can be put under the form $\phi^{\epsilon} x_0$, and that the congruence

$$\epsilon \equiv 0 \ (\text{mod. } p)$$

must not be satisfied (otherwise $fx=0$ would have equal roots), we see that

$$\chi_1 = \psi\phi^{\epsilon} x_0 = \phi^{q+\epsilon} x_0.$$

But this would indicate that χ_1 was a periodic function of some order other than the p^{th}, contrary to the hypothesis.

Hence the p functions

$$x_0,\ \psi x_0,\ \psi^2 x_0,\ \ldots \psi^{p-1} x_0$$

may be regarded as corresponding to the uniform functions

$$\chi_0,\ \chi_1,\ \chi_2,\ \ldots \chi_{p-1}.$$

24. Again, let

$$\phi^p = \pi,$$

then all the m roots of $fx=0$ are contained in the q rows

$$\begin{array}{llll} x_0, & \psi x_0, & \psi^2 x_0, & \ldots \psi^{p-1} x_0, \\ \pi x_0, & \psi\pi x_0, & \psi^2\pi x_0, & \ldots \psi^{p-1}\pi x_0, \\ \pi^2 x_0, & \psi\pi^2 x_0, & \psi^2\pi^2 x_0, & \ldots \psi^{p-1}\pi^2 x_0, \\ \vdots & \vdots & \vdots & \quad \vdots \\ \pi^{q-1} x_0, & \psi\pi^{q-1} x_0, & \psi^2\pi^{q-1} x_0, & \ldots \psi^{p-1}\pi^{q-1} x_0: \end{array}$$

for the supposition that two of the terms in the above rows were equal would lead to a relation of the form

$$\psi^{\alpha}\pi^{\beta} = \psi^{\gamma}\pi^{\delta}, \text{ or } \phi^{\alpha q+\beta p} = \phi^{\gamma q+\delta p},$$

or
$$\alpha q + \beta p = \gamma q + \delta p,$$
or
$$\frac{p}{q} = \frac{\alpha - \gamma}{\delta - \beta},$$
which cannot be, inasmuch as p and q are primes, and α and γ less than p, and β and δ than q. In like manner all the roots are comprised in the p rows

$$\begin{array}{llll} x_0, & \pi x_0, & \pi^2 x_0, & \ldots \pi^{q-1} x_0, \\ \psi x_0, & \pi\psi x_0, & \pi^2\psi x_0, & \ldots \pi^{q-1}\psi x_0, \\ \vdots & \vdots & \vdots & \vdots \\ \psi^{p-1} x_0, & \pi\psi^{p-1} x_0, & \pi^2\psi^{p-1} x_0, & \ldots \pi^{q-1}\psi^{p-1} x_0. \end{array}$$

The symbols ψ and π are, of course, commutative.

25. It follows that if the two roots $\phi^a x_0$ and $\phi^b x_0$ can be represented by uniform functions χ, then, if the congruence
$$b - a \equiv 0 \pmod{r}$$
is not satisfied when $r = p$ or $r = q$, all the roots may be represented by uniform functions which we may denote by
$$\chi_0, \ \chi_1, \ \chi_2, \ \cdots \chi_m,$$
each being derivable from another by the species of operation illustrated in Arts. 20 and 21.

26. If the last congruence is satisfied by $r = p$, or $r = q$, the roots may be distributed into $\frac{m}{r}$ sets, whereof each member is derivable from another by the species of operation referred to in the last article. If $r = p$ the sets may be denoted by

$$\begin{array}{llll} \chi_0, & \pi\chi_0, & \pi^2\chi_0, & \ldots \pi^{q-1}\chi_0, \\ \chi_1, & \pi\chi_1, & \pi^2\chi_1, & \ldots \pi^{q-1}\chi_1, \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{p-1}, & \pi\chi_{p-1}, & \pi^2\chi_{p-1}, & \ldots \pi^{q-1}\chi_{p-1}; \end{array}$$

while, if $r = q$, the sets may be denoted by

$$\begin{array}{llll} \chi_0, & \psi\chi_0, & \psi^2\chi_0, & \ldots \psi^{p-1}\chi_0, \\ \chi_1, & \psi\chi_1, & \psi^2\chi_1, & \ldots \psi^{p-1}\chi_1, \\ \vdots & \vdots & \vdots & \vdots \\ \chi_{q-1}, & \psi\chi_{q-1}, & \psi^2\chi_{q-1}, & \ldots \psi^{p-1}\chi_{q-1}. \end{array}$$

In the former case we obtain the series

$$\chi_0,\ \chi_1,\ \chi_2,\ \cdots \chi_{p-1},$$

and in the latter

$$\chi_0,\ \chi_1,\ \chi_2,\ \cdots \chi_{q-1};$$

then, finding ϕ_d as described in Art. 20, we obtain in the respective cases,

$$\pi = \phi_d^p,\quad \psi = \phi_d^q,$$

and are thus enabled to complete the distribution into sets. We observe, too, that the functional equations

$$\pi = \psi^{\frac{p}{q}},\quad \psi = \pi^{\frac{q}{p}}$$

express the relations between π and ψ, and are always soluble by the intervention of ϕ_d. The functions ϕ, π, and ψ are applied either to the roots or to the coefficients of the equation. But the function χ, by which abridged notation is represented the function

$$\chi\{I, J, \ldots L\},$$

is always a function of the coefficients of $fx = 0$.

27. The foregoing discussion includes the cases in which

$$m = 4,\quad m = 6,\quad m = 10,\quad m = 14,\quad m = 22, \text{ \&c.}$$

Those in which

$$m = 8,\quad m = 12,\quad m = 16,\quad m = 18, \text{ \&c.}$$

require further consideration. If the next step in the theory of equations be the discovery of the explicit forms of the transcendents, if any exist, by which the roots of equations of the sixth degree are expressed, the above investigation enables us to anticipate some of the properties of such transcendents.

4, Pump Court, Temple, London,
January 24, 1862.

SIR WM. HAMILTON'S ICOSIAN GAME.

By A. S. Herschel.

The pentagonal dodecahedron is the hemihedral form of the four-faced cube, so that if the six faces of a cube be covered by low ridges instead of by low pyramids on square bases, the pentagonal dodecahedron will be obtained when the ridges are parallel to each other in opposite pairs, but the three pairs themselves transverse to each other. Thus the pentagonal dodecahedron is formed by placing a pent roof upon each square face of a cube, the ends of every pent roof being flush with the slant sides of the adjacent roofs. This figure has therefore $8 + 6 \times 2 = 20$ angular points and $6 \times 5 = 30$ edges. As in the tetrahedron and the cube, the angular points are all tristigmatic, 3 edges meeting at every solid angle of the figure. A traveller therefore arriving by one edge at any angle must turn either to his right or to his left in leaving it by another edge, and if these operations be indicated respectively by the symbols $\lambda\mu$, it is required in the first problem of Sir Wm. Hamilton's icosian game to assign the order in which these operations must succeed each other, in order that twenty of them shall neutralize each other, and after visiting all the angles of the figure the traveller shall return at length to his original starting place. In the tetrahedron the only complete cinctures of the figure of this kind are the right-handed and left-handed cinctures $\lambda\mu\lambda$ and $\mu\lambda\mu$; in the cube $\lambda\lambda\mu\mu\lambda\lambda$ and $\mu\mu\lambda\lambda\mu\mu$; and in the pentagonal dodecahedron

$$\lambda\lambda\lambda\mu\mu\mu\lambda\mu\lambda\mu\lambda\lambda\lambda\mu\mu\mu\lambda\mu\lambda\mu \text{ and } \mu\mu\mu\lambda\lambda\lambda\mu\lambda\mu\lambda\mu\mu\mu\lambda\lambda\lambda\mu\lambda\mu\lambda.$$

If these cinctures be born in mind any permutation of the symbols $\lambda\mu$, 3 together being proposed, it will be easy to complete the cycle by starting from any place where this combination occurs in the cinctures.

GEOMETRICAL CURIOSITY.

By A. S. Herschel.

Two right-angled triangles have a common vertical side of 4 feet, and their other two horizontal sides respectively 2 feet and 3 feet. These are closed together until the latter two sides are the hypotenuse and base of a third right-angled triangle. Prove that the 3 facial angles at the apex of the tetrahedron so formed are together equal to a right angle.

Proof. Let ABC, ABD (fig. 40) be the original triangles; of which ABD (having sides $AB=4$, $DB=3$, $DA=5$) is thrown back upon the same plane with ABC whose sides $AB=4$, $BC=2$, and likewise the triangle ACd upon the same plane, which take for that of the paper. Then

$$Ad=AD=5=DC.$$

Bisect Ad in E and join ED, EC; E is the centre of the semi-circle which contains the right angle ACd, for ACd is necessarily a right angle, Cd having been drawn in a horizontal, perpendicular to BC in the vertical, plane ABC, and therefore perpendicular to AC.

Therefore by equal triangles

$$\angle ADE=\angle EDC,$$

and

$$ADE=\tfrac{1}{2}ADC,$$

$$\tan ADE=\sqrt{\left(\frac{1-\cos ADC}{1+\cos ADC}\right)}=\sqrt{\left(\frac{1-\frac{3}{5}}{1+\frac{3}{5}}\right)}$$

$$=\sqrt{(\tfrac{2}{8})}=\tfrac{1}{2}=\frac{AE}{AD};$$

therefore DAE is a right angle. Q.E.D.

If then upon a card-board ruled in squares, diagonals of oblongs be drawn through the points a, a, as in fig. 41, it will be seen that four such figures as the above will be formed which will fold upon the central rhombus $abab$, into a four-celled capsule that may be bound close by an elastic band thrown in a figure of 8 over its pointed extremities. Lids may be appended to the cells at c, c, c, c, and four such capsules will bind together conveniently into an octahedron of 16 cells.

ON THE DETERMINATION OF THE FOCI OF A CONIC.

By Rev. George Salmon.

When a conic is discussed by its trilinear equation, all problems necessarily assume their projective generality. Thus the problem to find the centre is identical with that of finding the pole of the line $x \sin A + y \sin B + z \sin C$: that of finding the condition that the equation should represent a parabola is identical with that of finding the condition that the same line should touch the curve. So the foci being the points of intersection of tangents through the two circular points at infinity, the problem of finding the foci in no respect differs from that of finding the intersections of tangents through any other two points. The method of solving the general problem being obvious, I had not cared to apply it to the particular case of the foci, till I was led to do so by Mr. Hensley's and Mr. Cayley's papers in the last two numbers. Some of the results, I think, are interesting.

Let the equation of the conic be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \text{ or } U = 0.$$

Let the condition that the line $\alpha x + \beta y + \gamma z$ may touch the conic (which we may call the tangential equation of the conic) be

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta = 0 \text{ or } \Phi = 0,$$

where $\qquad A = bc - f^2$, &c., $\quad F = gh - af$, &c.

Let the discriminant be called K:

$$K = abc + 2fgh - af^2 - bg^2 - ch^2.$$

Now if we are given one of the three pairs of lines which can be drawn through any four points on the curve, and it be required to find the other two pairs, the well known method is to equate to nothing the discriminant of

$$U + \lambda (\alpha' x + \beta' y + \gamma' z)(\alpha'' x + \beta'' y + \gamma'' z);$$

when we have a quadratic to determine λ; either root of which substituted in the form just written, causes it to represent a pair of the lines required.

Reciprocally; if we are given $x'y'z'$, $x''y''z''$, two of the six points in which any four tangents intersect, and if it be required to find the remaining points; we equate to nothing the discriminant of

$$\Phi+\lambda\,(x'\alpha+y'\beta+z'\gamma)\,(x''\alpha+y''\beta+z''\gamma),$$

when we have a quadratic to determine λ; either root of which substituted in the preceding brings it to the form

$$(x'''\alpha+y'''\beta+z'''\gamma)\,(x''''\alpha+y''''\beta+z''''\gamma),$$

where $x'''y'''z'''$, $x''''y''''z''''$ are the points required.

To apply this to the foci, let us observe that since in ordinary coordinates the line $\alpha x+\beta y+\gamma$ passes through one of the circular points at infinity when $\alpha^2+\beta^2=0$, the perpendicular on a line through one of these points, let fall from any point whatever will be infinite. By equating to nothing therefore the denominator in the expression for the length of a perpendicular in trilinear coordinates, we learn that the condition that $\alpha x+\beta y+\gamma z$ may pass through either of the circular points at infinity is

$$\alpha^2+\beta^2+\gamma^2-2\beta\gamma\cos A-2\gamma\alpha\cos B-2\alpha\beta\cos C=0 \text{ or } \psi=0.$$

We may call this the tangential equation of the two circular points; and it may be noted in passing that the condition that two lines should be mutually perpendicular is

$$\alpha'\,\frac{d\psi}{d\alpha}+\beta'\,\frac{d\psi}{d\beta}+\gamma'\,\frac{d\psi}{d\gamma}=0,$$

the reason for which form is sufficiently evident.

Let us now form the discriminant of $\Phi+\lambda\psi$, and the result is

$$K^2+\lambda KP+\lambda^2Q=0,$$

where $P=0$ is the condition that the equation may represent an equilateral hyperbola:

$$P=a+b+c-2f\cos A-2g\cos B-2h\cos C,$$

and $Q=0$ is the condition that the equation may represent a parabola:

$$Q=A\sin^2A+B\sin^2B+C\sin^2C$$
$$+2F\sin B\sin C+2G\sin C\sin A+2H\sin A\sin B.$$

$P^2=4Q$ is the condition that the curve may pass through either of the circular points at infinity; and will of course

be satisfied when the curve passes through both; that is to say, is a circle. This latter condition can in various ways be resolved into the sum of two squares, both of which vanish separately when the curve is a circle; for example,

$$\{a\cos 2\alpha + b\cos 2\beta + c\cos 2\gamma + 2f\cos(\beta+\gamma) + 2g\cos(\gamma+\alpha) + 2h\cos(\alpha+\beta)\}^2$$
$$+ \{a\sin 2\alpha + b\sin 2\beta + c\sin 2\gamma + 2f\sin(\beta+\gamma) + 2g\sin(\gamma+\alpha) + 2h\sin(\alpha+\beta)\}^2,$$

where α, β, γ are the angles made with any axis by perpendiculars on the sides of the fundamental triangle.

Having determined λ from the preceding quadratic, either root substituted in $\phi+\lambda\psi$ reduces it to the form

$$(\alpha x' + \beta y' + \gamma z')(\alpha x'' + \beta y'' + \gamma z''),$$

where $x'y'z'$, $x''y''z''$ are the coordinates of two of the foci. For one value of λ the factors of $\phi+\lambda\psi$ are real; for the other they are imaginary. I believe this to be the easiest way in practice of determining the coordinates of the foci, when the conic is given by an equation in rectangular Cartesian coordinates, with numerical coefficients. The quadratic for λ then is

$$K^2 + K(a+b)\lambda + (ab-h^2)\lambda^2 = 0.$$

Thus let the equation be

$$2x^2 + 4xy - y^2 + 24y + 6 = 0,$$

we have $K=-324$; and the values of λ are 108 and -162. The equation ϕ is

$$-150\alpha^2 + 12\beta^2 - 6\gamma^2 - 48\beta\gamma + 48\gamma\alpha - 24\alpha\beta = 0.$$

Giving λ the value 108, $\phi+\lambda(\alpha^2+\beta^2)$ breaks up into the factors $\alpha+2\beta-\gamma$, $-7\alpha+10\beta+\gamma$, shewing that the coordinates of the two real foci are -1, -2; and -7, 10. Giving λ the value -162; the factors are

$$-\{4+6\sqrt{(-1)}\}\alpha + \{4-3\sqrt{(-1)}\}\beta+\gamma$$
$$-\{4-6\sqrt{(-1)}\}\alpha + \{4+3\sqrt{(-1)}\}\beta+\gamma,$$

which give the coordinates of the imaginary foci.

Returning now to the general problem; we find the equation of all conics confocal to a given one by forming the reciprocal of $\phi+\lambda\psi$, when the result is

$$KU + \lambda S + \lambda^2(x\sin A + y\sin B + z\sin C)^2.$$

The coefficient of λ^2 is constant: and the coefficient of λ equated to zero denotes the circle locus of the intersection of tangents to U, which are at right angles to each other.

$$\begin{aligned}S=(B+C+2F\cos A)x^2+(C+A+2G\cos B)y^2+(A+B+2H\cos C)z^2\\ -2(F+G\cos C+H\cos B-A\cos A)yz\\ -2(G+H\cos A+F\cos C-B\cos B)zx\\ -2(H+F\cos B+G\cos A-C\cos C)xy.\end{aligned}$$

From the fact that the function S is of the first degree in the coefficients of the reciprocal conic, immediately follows a theorem lately given by Mr. Ferrers; viz. that if a conic touch four fixed lines, the circle S passes through two fixed points.

The reciprocal of $\phi+\lambda\psi$, when we give to λ either value found by solving the preceding quadratic, denotes the square of the equation of one of the axes of the curve. The axes may also be found by forming the determinant

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dS}{dx}, & \frac{dS}{dy}, & \frac{dS}{dz} \\ \sin A, & \sin B, & \sin C \end{vmatrix},$$

which will give the product of the equation of the axes by the constant factor $x\sin A+y\sin B+z\sin C$.

When the curve is a parabola, Q vanishes; the quadratic reduces to a simple equation; and $P\phi-K\psi$ resolves itself into the two factors which give the foci.

The preceding principles may also be applied to tetrahedral coordinates. The tangential equation of the circle at infinity is as before

$$\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\beta\cos(ab)-2\alpha\gamma\cos(ac)-2\alpha\delta\cos(ad)$$
$$-2\beta\gamma\cos(bc)-2\beta\delta\cos(bd)-2\gamma\delta\cos(cd),$$

where (ab) denotes the angle between two of the planes of reference. Let U be the equation of any quadric, ϕ its tangential equation, or the condition that $\alpha x+\beta y+\gamma z+\delta w$ may touch the surface; then we get the equation of all surfaces confocal to the given one by forming the reciprocal of

$\phi + \lambda\psi$, where ψ is the equation just given of the circle at infinity.

The reciprocal is of the form $K^2U + \lambda S + \lambda^2 S' + \lambda^3 t$. The coefficient of λ^3 is constant: S' denotes the sphere which is the locus of the intersection of three tangent planes at right angles; and S denotes the locus of points whence three tangent lines are at right angles. This is a particular case of the following: If from any point three tangent planes to a quadric U be a conjugate system with respect to another V, then from the same point three tangent lines to V will be conjugate with respect to U. If ϕ and ψ equated to nothing respectively be the tangential equations of the two quadrics, the reciprocal of $\phi + \lambda\psi$ will be

$$K^2U + \lambda S + \lambda^2 S' + \lambda^3 K'^2 V,$$

where S is the locus of points whence three tangent planes to V are conjugate with respect to U and S' of those whence three tangent planes to U are conjugate with respect to V. If we form the discriminant of $\phi + \lambda\psi$, solve the resulting cubic, and substitute the resulting value in $\phi + \lambda\psi$, we get the tangential equations of the polar conics of the given quadric.

Since S' is of the first degree in the coefficients of the reciprocal of U, it follows that if a quadric touch eight fixed planes, the sphere, which is the locus of the intersection of three rectangular tangent planes, passes through a fixed circle; or, more generally, that the covariant quadric S', taken with regard to any fixed quadric V, passes through a fixed curve.

Trinity College, Dublin,
June 25, 1862.

ON THE RADICAL AXIS OF TWO SIMILAR AND SIMILARLY SITUATED CONICS.

By N. M. Ferrers.

In the curve

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

let d_1, d_2, d_3 be the diameters parallel to the lines of reference; then we know that

$$\frac{v + w - 2u'}{d_1^2} = \frac{w + u - 2v'}{d_2^2} = \frac{u + v - 2w'}{d_3^2}.$$

Let each of these fractions $= \lambda$. Then the equation of the curve may be written

$$u\alpha^2 + v\beta^2 + w\gamma^2 + (v + w)\beta\gamma + (w + u)\gamma\alpha + (u + v)\alpha\beta$$
$$= \lambda(d_1^2\beta\gamma + d_2^2\gamma\alpha + d_3^2\alpha\beta),$$

or $\quad (u\alpha + v\beta + w\gamma)(\alpha + \beta + \gamma) = \lambda(d_1^2\beta\gamma + d_2^2\gamma\alpha + d_3^2\alpha\beta).$

Hence $\quad u\alpha + v\beta + w\gamma = 0$

is the equation of the chord of intersection of the given curve with a similar and similarly situated one, described about the triangle of reference.

Let $\quad p\alpha^2 + q\beta^2 + r\gamma^2 + 2p'\beta\gamma + 2q'\gamma\alpha + 2r'\alpha\beta = 0$

be another curve similar and similarly situated to the given one. Let its equation be written, by a transformation similar to the above,

$$(p\alpha + q\beta + r\gamma)(\alpha + \beta + \gamma) = \mu(d_1^2\beta\gamma + d_2^2\gamma\alpha + d_3^2\alpha\beta).$$

Then, the chord of intersection of these two curves will be given by the equation

$$\frac{u\alpha + v\beta + w\gamma}{\lambda} = \frac{p\alpha + q\beta + r\gamma}{\mu}.$$

Now,

$$\lambda = \frac{v+w-2u'}{d_1^2} = \frac{w+u-2v'}{d_2^2} = \frac{u+v-2w'}{d_3^2} = 2\,\frac{u+v+w-u'-v'-w'}{d_1^2+d_2^2+d_3^2},$$

similarly, $\quad \mu = 2\,\dfrac{p + q + r - p' - q' - r'}{d_1^2 + d_2^2 + d_3^2}.$

Hence the common chord (or radical axis) of the two given curves is represented by the equation

$$\frac{u\alpha + v\beta + w\gamma}{u + v + w - u' - v' - w'} = \frac{p\alpha + q\beta + r\gamma}{p + q + r - p' - q' - r'}.$$

COR. The triangular equation of the inscribed circle being

$$(s-a)^2\alpha^2 + \ldots - 2(s-b)(s-c)\beta\gamma - \ldots = 0,$$

and of the six-points circle,

$$(b^2 + c^2 - a^2)\alpha^2 + \ldots - 2a^2\beta\gamma - \ldots = 0,$$

that of their radical axis will be found from above to be

$$(a-b)(a-c)\alpha + \ldots = 0,$$

which may easily be shewn to be a tangent to either.

Aug. 31, 1861.

ON THE EQUATION OF THE SIX-POINTS CIRCLE.

By HENRY R. GREER, A.B.

THE equation of the circle circumscribing the triangle

$$x \sin A + y \sin B - z \sin C = 0;$$
$$x \sin A - y \sin B + z \sin C = 0;$$
$$-x \sin A + y \sin B + z \sin C = 0;$$

that is, of the circle passing through the middle points of the sides of the triangle of reference, after some reduction, the equation comes out

$$x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C$$
$$- 2(yz \sin A + zx \sin B + xy \sin C) = 0 \ldots\ldots(1).$$

In like manner, we can form the equation of the circle circumscribing the triangle

$$x \cos A + y \cos B - z \cos C = 0;$$
$$x \cos A - y \cos B + z \cos C = 0;$$
$$-x \cos A + y \cos B + z \cos C = 0;$$

and its equation will be found, after some reduction, to be identical with (1). This is therefore the equation of the six-points circle, and in fact it can be easily verified that it

passes through the middle points of the sides and through the feet of the perpendiculars of the triangle ABC. Denoting, as usual, by a, b, c the lengths of the sides, (1) may be written in the form

$$x^2.a\cos A+y^2.b\cos B+z^2.c\cos C-(ayz+bzx+cxy)=0\ldots(\text{i}).$$

The equation of the inscribed circle is

$$a^2(s-a)^2x^2+b^2(s-b)^2y^2+\ldots$$
$$-2bc(s-b)(s-c)yz=0 \;\ldots\ldots\ldots\ldots\; (2).$$

The condition that (1) and (2) should touch, is the condition that their radical axis should touch either of them. The method of finding the radical axis of two circles has been indicated by Mr. Slesser (*Quarterly Journal*, Vol. IV., p. 135). We must identify the expression $S-\theta S'$ with $(ax+by+cz)(\lambda x+\mu y+\nu z)$, then $\lambda x+\mu y+\nu z=0$ is the radical axis of S and S'. Applying this method to (i) and (2), we easily find the equation of their radical axis to be

$$aAx+bBy+cCz=0\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

where A denotes $ab+ac-a^2-bc$, or $(c-a)(a-b)$,

B .... $bc+ba-b^2-ca$, or $(a-b)(b-c)$,

C .... $cb+ca-c^2-ab$, or $(b-c)(c-a)$,

The condition that (3) should touch (i) is, after some simplification, that the determinant

$$\begin{vmatrix} b^2+c^2-a^2, & -c^2, & -b^2, & A \\ -c^2, & c^2+a^2-b^2, & -a^2, & B \\ -b^2, & -a^2, & a^2+b^2-c^2, & C \\ A, & B, & C, & 0 \end{vmatrix} \text{ should } = 0.^*$$

I must confess that I have not been able to discover any *à priori* method for evaluating this determinant. However it is not troublesome to work it out, when it will be found to be identically zero, observing that $B+C=-(b-c)^2$, and similarly for the other two, and that $BC+CA+AB=0$ identically.

Similar proofs of course hold for the tangency of the six-points circle with any of the escribed circles of the original triangle.

* See Ferrers' *Trilinear Coordinates*, p. 75.

If a circle be circumscribed to ABC, and tangents to it drawn at the three vertices, these will intersect the opposite sides in three points lying in one right line which call X. If, of the circle inscribed in ABC the points of contact with the sides be joined each with the opposite vertex, and at the points where these lines meet the circle again tangents to it be drawn, they will intersect the opposite sides in three points lying in one right line which call Y. The equations of X and Y are

$$bcx + cay + abz = 0,$$

and $$a(s-a)x + b(s-b)y + c(s-c)z = 0,$$

respectively.* Hence we see at once that the common tangent to the six-points circle and the inscribed circle passes through the intersection of X and Y; analogous theorems obtaining for each of the escribed circles also.

In Mr. Casey's paper on the six-points circle (*Quarterly Journal*, Vol. IV., p. 245) there is given Sir Wm. Hamilton's theorem, viz. if P be the intersection of the perpendiculars of the triangle ABC, the six-points circle of ABC is also the six-points circle of BPC, &c. This is obvious, since the feet of the perpendiculars of ABC are the same points as the feet of the perpendiculars of BPC, &c. Mr. Casey's theorem, given in § 11 of his paper, may be most simply proved by observing that the six-points circle of ABC is the circle circumscribing the triangle formed by joining the feet of the perpendiculars of ABC, and of *this* triangle the centres of its inscribed and escribed circles are the points P, A, B, C. This triangle (formed by joining the feet of the perpendiculars) is, of course, quite as *general* a triangle as the original triangle. If L, M, N be the middle points of the sides of ABC, then the bisectors of the angles of the triangle LMN are the radical axes of the inscribed and escribed circles of ABC, a theorem given by Mr. Casey, § 12. To prove this theorem by trilinear coordinates; take, for example, the inscribed circle and that escribed to the side a. The equation of the former has been given before, (2); that of the latter is

$$a^2s^2(s-a)^2x^2 + b^2(s-c)^2(s-a)^2y^2 + c^2(s-a)^2(s-b)^2z^2$$
$$-2bc(s-a)^2(s-b)(s-c)yz + 2ca(s-a)^2s(s-b)zx$$
$$+2ab(s-a)^2s(s-c)xy = 0.$$

* See Salmon's *Conic Sections*, Chap. VIII.

Putting $$a^2s^2(s-a)^2-\theta a^2(s-a)^2=a\lambda,$$
&c.
$$-2bc(s-a)^2(s-b)(s-c)+2\theta bc(s-b)(s-c)=b\nu+c\mu,$$
&c.,

we readily find $\theta=(s-a)^2$, and consequently
$$\lambda=a^2(s-a)^2(b+c),$$
and similarly for μ and ν; therefore the equation of the required radical axis is
$$a(b+c)x+b(b-c)y-c(b-c)z=0.$$

Now the equation (reduced to the form a) of LM is $\frac{ax+by-cz}{2c}=0$; and of LN, $\frac{ax-by+cz}{2b}=0$; and the equation of the external bisector of the angle between these two lines is identical with that of the radical axis given above. I may remark that Mr. Casey's proof of the theorem of the six-points circle may be somewhat simplified at two important points; first, by shewing (what can easily be done) that the centre of the six-points circle may be found by bisecting the intervals between the feet of the perpendiculars of the triangle ABC and the middle points of its sides, and then drawing perpendiculars at these points to the sides; whence it follows at once that the centre of Σ lies half-way between P and the centre of X; *i.e.* that "P is the external centre of similitude of X and Σ:" secondly, by observing that the triangle FGH may be constructed from Σ and Δ (the triangle formed by joining the middle points of the sides of ABC) just as AED is constructed from X and ABC, whence it follows that these triangles are similar; and a pair of corresponding sides in each (GH and ED) being parallel, the remaining sides are parallel, and therefore that "FG is perpendicular to AD, and FH to AE."

Royal Military College,
26 *May*, 1862.

NOTE ON THE INCLINATION OF THE OPTIC AXIS TO THE RAY AXIS OF A BIAXAL CRYSTAL.

By WILLIAM WALTON, M.A., Trinity College.

THE object of this note is to prove the following property respecting the angle between the two axes:

"The greatest and least of the optic constants being assigned, the angle between the optic and ray axes will be a maximum, when the optic constant of intermediate magnitude is a mean proportional between the greatest and least of the optic constants."

The direction-cosines of the optic axis are equal to

$$\left(\frac{a^2-b^2}{a^2-c^2}\right)^{\frac{1}{2}},\ 0,\ \left(\frac{b^2-c^2}{a^2-c^2}\right)^{\frac{1}{2}}:$$

those of the ray-axis are equal to

$$\left(\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)^{\frac{1}{2}},\ 0,\ \left(\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{1}{a^2}-\frac{1}{c^2}}\right)^{\frac{1}{2}}:$$

hence, if θ be the angle between them,

$$\cos\theta = \frac{c}{b}\cdot\frac{a^2-b^2}{a^2-c^2}+\frac{a}{b}\cdot\frac{b^2-c^2}{a^2-c^2}$$

$$=\frac{(ca+b^2)(a-c)}{b(a^2-c^2)}=\frac{ca+b^2}{b(a+c)}.$$

Considering a and c constant, and b variable, it is easily seen that $\cos\theta$ is a minimum and therefore θ a maximum when $b=(ca)^{\frac{1}{2}}$. The corresponding value of $\cos\theta$ is given by the equation

$$\cos\theta=\frac{2}{\left(\frac{c}{a}\right)^{\frac{1}{2}}+\left(\frac{a}{c}\right)^{\frac{1}{2}}}.$$

September 4, 1861.

PROPERTIES OF THE EIGHT CIRCLES WHICH ARE TANGENTIAL TO THREE GIVEN CIRCLES.

By John Casey, Scholar of Trinity College, Dublin, and Science-Master, Kingstown Schools, Kingstown.

1. Let ABC (fig. 42) *be any triangle and* $B'C'$ *a line drawn parallel to the base* BC. *Then if* O, O' *be the escribed circles to the triangle* ABC *opposite the angles* B *and* C *respectively,* O_1 *the escribed circle of the triangle* $AB'C'$ *opposite the angle* A, *and,* O_1' *its escribed circle, I say besides the lines* AB, AC, *the four circles* O, O', O_1, O_1' *are each touched by two other circles.*

Demonstration. Bisect BC in D, draw DH perpendicular to the line joining the centres of O and O'. Join AD, produce it till it meets $B'C'$ in D', produce OO' to meet BC produced in I. Join O_1O_1' which evidently passes through A, and from D and D' draw DE, $D'E'$ perpendicular to O_1O_1', and let DH produced meet $D'E'$ in G, from G draw GL, GL' perpendicular to BC, $B'C'$.

Now from similar triangles

$$AI : DE, \text{ or } GE' :: AF : FE :: AF' : F'E'.$$

Hence the points G, F', I are in a right line. Again, since A and I are the centres of similitude of O and O', and A and F' the centres of similitude of O_1 and O_1', the circles described on AI and AF' are respectively coaxial with O, O' and with O_1, O_1', and it is evident that GH is the radical axis of O, O' and GE' of $O_1 : O_1'$; hence if R, R' be the radii of the circles whose common centre is G and which cut orthogonally the coaxial systems O, O', and O_1, O_1' respectively, it is evident that $R^2 = IG.GK$ and $R'^2 = F'G.GK$ [AK being drawn perpendicular to IG]; therefore

$$R^2 : R'^2 :: IG : F'G :: GL : GL',$$

hence
$$\frac{R^2}{GL} = \frac{R'^2}{GL'}.$$

Therefore the circle inverse to the line BC with respect to the circle whose radius is R coincides with the circle in-

verse to the line $B'C'$ with respect to the circle whose radius is R' and since BC touches the circles O, O', and $B'C'$ touches O', O_1' the inverse circle evidently touches O, O', O_1, O_1'.

Again, through I draw a second common tangent to O, O' and through F' a second common tangent to O_1, O_1', and by means of these tangents, since they are evidently parallel, it may be proved in exactly the same manner that another circle touches the four circles O, O', O_1, O_1', hence the proposition is proved. Q.E.D.

COR. Hence we derive the following construction for the points of contact of a circle touching the inscribed and escribed circles of a plane triangle.

Let ABC (fig. 43) *be the triangle, draw LM a second common tangent to the circles O, O' and L_1M_1 to O_1, O_1', then if the middle point of AB be joined to the points of contact of these tangents, the second points of intersection of the joining lines with the circles will determine the points required.* (See Dublin Examination Papers, 1862, Sizarship Questions).

2. By inversion from any auxiliary point in the plane of the triangle, we have the following proposition (Dr. Hart's "Extension of Terquem's Theorem," see *Quarterly Journal of Pure and Applied Mathematics*, Vol. IV.):

If ABC (fig. 44) *be a triangle whose sides are formed by arcs of circles, and if circles O, O', O'', O''' corresponding to the inscribed and escribed circles of a plane triangle be described, these four circles are tangential to a fourth circle Σ besides the sides of the triangle ABC.*

3. Since, in fig. 42, GH is the radical axis of O, O', and GE' of O_1, O_1', and GK of the two circles which are proved in Art. 1 to touch the circles O, O', O_1, O_1', and fig. 44 is derived from fig. 42 by inversion. Hence we have the following theorem:

If (fig. 44) *through the point A' three circles be described coaxial with O, O', O'', O''', Σ, BCB' respectively; these circles so described are coaxial.*

4. Since the circle described upon AI (fig. 42) is coaxial with O, O', and the circle described upon AF' coaxial with O_1, O_1'. Hence we have, as in Art. 3, the following theorem:

If (fig. 44) *through the point A three circles be described coaxial with O, O', O''; O''', Σ, BCB' respectively; these circles are coaxial.*

5. Since $B'C'$, in fig. 42, is parallel to BC; hence, in fig. 44, we have the following theorem:

If through the point A' two other circles besides AB and AC be described touching O, O', and two touching O'', O'''; these four circles so described form two pair of circles tangential to each other.

6. If C, C', C'' be any three circles, it is easy to see the eight circles tangential to them may be divided into groups of four circles, each in the following manner:

$$\begin{array}{ll}
\alpha,\ \beta,\ \gamma,\ \delta; & \alpha',\ \beta',\ \gamma',\ \delta'; \\
\alpha,\ \beta,\ \gamma,\ \delta'; & \alpha',\ \beta',\ \gamma',\ \delta; \\
\alpha,\ \beta,\ \gamma',\ \delta; & \alpha',\ \beta',\ \gamma,\ \delta'; \\
\alpha,\ \beta',\ \gamma,\ \delta; & \alpha',\ \beta,\ \gamma',\ \delta'; \\
\alpha',\ \beta,\ \gamma,\ \delta; & \alpha,\ \beta',\ \gamma',\ \delta'; \\
\alpha,\ \beta,\ \gamma',\ \delta'; & \alpha',\ \beta',\ \gamma,\ \delta; \\
\alpha,\ \beta',\ \gamma,\ \delta'; & \alpha',\ \beta,\ \gamma',\ \delta; \\
\alpha',\ \beta,\ \gamma,\ \delta'; & \alpha,\ \beta',\ \gamma',\ \delta;
\end{array}$$

each group standing in the second column being circles inverse to those in the corresponding group in the first column with respect to the circle which cuts C, C', C'' orthogonally, and also it is evident that the circles in each group are tangential to a fourth circle besides C, C', C''. Hence we have the following theorem:

The eight circles which are tangential to three given circles may be divided in eight different ways into two groups of four circles each, such that the four circles of each group are tangential to a fourth circle besides the three given circles.

7. Since the eight circles tangential to C, C', C'' may be divided into two groups α, β, γ, δ; α', β', γ', δ'; which are inverse to each other with respect to the circle which cuts C, C', C'' orthogonally, it is easy to see that any two pair of circles inverse to each other such as α, α'; β, β' have a common radical centre, hence each of the four circles α, α'; β, β' is cut in involution by the other three, and also the

radical axes of the four circles form a pencil in involution. Hence we have the following theorem:

The eight circles which are tangential to three given circles may be divided in six ways into groups of four circles in each, such that the radical axes of each group form a pencil in involution, and such that each circle is cut by the other three circles of the group to which it belongs in involution.

COR. *Since each circle belongs to six groups, the intersections of each circle, by the other seven circles of the tangential system, may be divided into six series of points, each series of which are in involution.*

8. *Properties of points of contact on C, C', C''.*

Let the points of contact of α, α' on the circles C, C', C'' be denoted by a, a_1; a', a_1'; a'', a_1'' respectively, and let the points of contact of the other circles be denoted similarly. Now if R be the radical centre of C, C', C'', it is easy to see that the rectangles

$$Ra.Ra_1,\quad Ra'.Ra_1',\quad Ra''.Ra_1'',\quad Rb.Rb_1,\ \&c.$$

are each equal to the square of the radius of the circle which cuts C, C', C'' orthogonally; hence it is easy to see *that any two pair of points a, a_1 and b, b_1 are on the circumference of a circle, and therefore that the twenty-four points of contact on C, C', C'', are four by four on forty-eight circles which may be divided into twelve groups of eight circles each which are coaxial, and that taking any four circles of the forty-eight which are not coaxial, it will be seen that any circle of the four is cut in involution by the other three circles, and the radical axes of the four circles form a pencil in involution, and moreover, any circle of the forty-eight, together with the circles C, C', C'', form a group of four circles, such that each circle is cut in involution by the other three and their radical axes form a pencil in involution.*

9. *Properties of the limiting points of the eight circles tangential to three given circles.*

It was proved in Art. 7 that any group of circles such as α, α'; β, β' have a common radical centre; hence the twelve limiting points of these circles all lie on the circumference of a circle cutting α, α'; β, β' orthogonally, and since the limiting points of the pairs of circles α, α'; α, β; α, β' lie on lines passing through the centre of α; hence it is easy to see that the twelve limiting points form systems of points

in involution, and taking the eight circles into account we have the following theorem:

The fifty-six limiting points of the eight circles which are tangential to three given circles are placed on the circumferences of six circles twelve by twelve, these six circles are coaxial three by three, their centres are the centres of similitude of the given circles C, C', C'', and the twelve limiting points on each may be divided into four groups of six points each, each group of which are in involution.

10. *Properties of the centres of the eight circles.*

Since four of the eight circles are the inverse of the other four, it is evident that the eight points of contact on any of the circles C, C', C'' are in involution; hence the anharmonic ratio of four of these points is equal to the anharmonic ratio of the other four homologous points. Again, if we take two of the circles, C, C' for instance, it is easy to see that the centres of the eight tangential circles lie four and four on two confocal conics, whose foci are the centres of C, C', and that the anharmonic ratio of the four points on one conic is equal to the anharmonic ratio of the four points on the other conic, since each ratio is equal to the anharmonic ratio of the four points of contact corresponding to them on the circles. Hence taking the three circles into account, we have the following theorem:

The centres of the eight circles which are tangential to three given circles lie in groups of four each on six conics which are confocal two by two, and the anharmonic ratio of the four points on any conic is equal to the anharmonic ratio of the four points on its confocal conic.

11. By inversion from an auxiliary point in space, we have the theorems proved in the previous articles from 2 to 10 inclusive, established for the surface of the sphere.

Kingstown Schools,
Kingstown,
February 1*st*, 1862.

NOTE ON THE m^{th} DIFFERENCES OF 0.

By GEORGE SCOTT, M.A., Trinity College, Dublin.

A COMPARISON of the two expressions which I have given Vol. IV. of this *Journal*, for $\frac{d^n}{dx}$ of $\phi(y)$, enables us to obtain the value of $\Delta^m 0^n$ more easily, in some instances, than by the ordinary method.

The labour of the latter method increases as m and n get larger; the formula, I am about to give, is comprised in a few terms, whether m and n be large or small, *provided that their difference be small*, it therefore accommodates itself to those cases, precisely, in which the ordinary method is most laborious.

The two expressions referred to are the following:

$$\frac{d^n}{dx} = \Sigma\left[\frac{1}{1.2\ldots m}\left\{\frac{d^n y^m}{dx^n} - \frac{m}{1}\,y\,\frac{d^n y^{m-1}}{dx^n} + \frac{m(m-1)}{1.2}\,\frac{d^n y^{m-2}}{dx^n}\right\}\frac{d^m}{dy}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and
$$\frac{d^n}{dx} = 1.2.3\ldots n\Sigma\left\{\frac{y_1^{a_1}}{a_1}\,\frac{y_2^{a_2}}{a_2}\ldots\frac{d^m}{dy}\right\} \quad \ldots\ldots\ldots(2).$$

where
$$y_s = \frac{\frac{d^s y}{dx^s}}{1.2.3\ldots r},$$

and
$$a_1 + a_2 + a_3 + \&c. = m,$$
$$a_1 + 2a_2 + 3a_3 + \&c. = n.$$

On examining the expanded form of the expression (2), given in Vol. IV., p. 83, it will be seen that the formula may be written thus:

$$\frac{d^n}{dx} = \Sigma\left\{n(n-1)\ldots m\left\{\begin{matrix} y_1^{m-1}y_{n-m+1} + \dfrac{(m-1)}{1.2}\,y_1^{m-2}S_2^{n-m+2} \\ + \dfrac{(m-1)(m-2)}{1.2.3}\,y_1^{m-3}S_3^{n-m+3} + \&c. \end{matrix}\right\}\frac{d^m}{dy^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where S_i^{n-m+i} means the sum of such *permutations* and *repetitions* of the functions y_1, y_2, y_3, y_4, &c., taken i of them together as make the sum of the suffixes equal to $n-m+i$.

Equating the coefficients of $\frac{d^m}{dy}$ in (1) and (3), we obtain an identity susceptible of useful applications

$$\frac{d^n y^m}{dx^n} - \frac{m}{1} y \frac{d^n y^{m-1}}{dx^n} + \frac{m(m-1)}{1.2} y^2 \frac{d^n y^{m-2}}{dx^n} - \&c.$$

$$= 1.2.3 \ldots n \left\{ m y_1^{m-1} y_{n-m+1} + \frac{m(m-1)}{1.2} y_1^{m-2} S_2^{n-m+2} + \&c. \ldots (4). \right.$$

The notation in the first term of the right-hand member is accurate only so long as n and m are different.

If we substitute ε^x for y, the left-hand member of the equation becomes $\varepsilon^{mx} \Delta^m 0^n$ and the right-hand member will be divisible by ε^{mx}. Denoting by $s_2^{n-m+2} s_3^{n-m+3}$, &c. what S_2^{n-m+2}, &c. become after the substitution, we get the following formula:

$$\Delta^m 0^n = 1.2.3 \ldots n \left\{ \frac{m}{1.2 \ldots n-m+1} + \frac{m(m-1)}{1.2} s_2^{n-m+2} + \&c. \right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

for calculating $\Delta^m 0^n$.

Ex. Let $\qquad m = n - 4,$

$$\Delta^{n-4} 0^n = 1.2 \ldots n \left\{ \begin{array}{l} \frac{n-4}{1.2.3.4.5} + \frac{(n-4)(n-5)}{1.2} \left\{ \frac{1}{2} \cdot \frac{1}{2.3.4} + \frac{1}{2.3.4} \cdot \frac{1}{2} + \frac{1}{2.3} \cdot \frac{1}{2.3} \right\} \\ + \frac{(n-4)(n-5)(n-6)}{1.2.3} \left\{ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2.3} + \frac{1}{2} \cdot \frac{1}{2.3} \cdot \frac{1}{2} + \frac{1}{2.3} \cdot \frac{1}{2} \cdot \frac{1}{2} \right\} \\ + \frac{(n-4) \ldots (n-7)}{1.2.3.4} \cdot \frac{1}{2^4}. \end{array} \right.$$

If $\qquad n = 10,$

$$\Delta^6 0^{10} = 2.3.4 \ldots 10 \left\{ \frac{1}{20} + 5 \left(\frac{1}{8} + \frac{1}{12} \right) + \frac{5}{2} + \frac{15}{16} \right\};$$

therefore $\qquad \Delta^6 0^{10} = 16435440,$

the value assigned to it in the table given by Professor De Morgan's *Calculus*, p. 253, and quoted by Dr. Boole (*Finite Differences*, p. 20).

Brighton Vale, Monkstown, Co. Dublin,
June 12th, 1862.

ON THE DISCOVERY AND PROPERTIES OF A PECULIAR CLASS OF ALGEBRAIC FORMULÆ.

By JOHN BLISSARD, M.A.

THE power of any new analytical method may be shewn, not only in the production of results which are strictly due to that method and can be obtained or proved in no other way, but also in the *origination* of formulæ which are subsequently found capable of proof by other methods, from which however they were never likely to have sprung. Instances of this kind will here be presented to the reader's notice, for although the leading formulæ about to be exhibited will be proved by means of common Algebra, I am indebted for their discovery to the use of the representative method and notation adopted and developed in my "Theory of Generic Equations."

1. Let C_n be any function of n, and let

$$\phi_1(n) = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n},$$

$$\phi_2(n) = \frac{\phi_1(1)}{C_1} + \frac{\phi_1(2)}{C_2} + \ldots + \frac{\phi_1(n)}{C_n},$$

$$\phi_3(n) = \frac{\phi_2(1)}{C_1} + \frac{\phi_2(2)}{C_2} + \ldots + \frac{\phi_2(n)}{C_n}, \ \ldots$$

$$\phi_{r+1}(n) = \frac{\phi_r(1)}{C_1} + \frac{\phi_r(2)}{C_2} + \ldots + \frac{\phi_r(n)}{C_n},$$

it is required to prove that, n being a positive integer,

$$\phi_r(n) = C_1C_2C_3\ldots C_n\left\{\frac{1}{(C_2-C_1)(C_3-C_1)\ldots(C_n-C_1)}\cdot\frac{1}{C_1^{r+1}}\right.$$

$$-\frac{1}{(C_2-C_1)(C_3-C_2)\ldots(C_n-C_2)}\cdot\frac{1}{C_2^{r+1}}$$

$$\left.+\frac{1}{(C_3-C_1)(C_3-C_2)(C_4-C_3)\ldots(C_n-C_3)}\cdot\frac{1}{C_3^{r+1}} - \&c.\right\}.$$

First, it is evident from the construction of the above set of equations that $\phi_r(n+1) - \phi_r(n)$ or $\Delta\phi_r(n) = \dfrac{\phi_{r-1}(n+1)}{C_{n+1}}$.

Next, assume

$$F_r(n) = C_1 C_2 \ldots C_n \left\{ \frac{1}{(C_2 - C_1)(C_3 - C_1)\ldots(C_n - C_1)} \cdot \frac{1}{C_1^{r+1}} \right.$$

$$\left. - \frac{1}{(C_2 - C_1)(C_3 - C_2)\ldots(C_n - C_2)} \cdot \frac{1}{C_2^{r+1}} + \&c. \right\},$$

then $F_r(n+1) = C_1 C_2 \ldots C_{n+1} \left\{ \frac{1}{(C_2 - C_1)(C_3 - C_1)\ldots(C_{n+1} - C_1)} \cdot \frac{1}{C_1^{r+1}} \right.$

$$\left. - \frac{1}{(C_2 - C_1)(C_3 - C_2)\ldots(C_{n+1} - C_2)} \cdot \frac{1}{C_2^{r+1}} + \&c. \right\},$$

hence $\Delta F_r(n)$ which $= F_r(n+1) - F_r(n)$

$$= C_1 C_2 \ldots C_n \left\{ \frac{1}{(C_2 - C_1)(C_3 - C_1)\ldots(C_n - C_1} \cdot \left(\frac{C_{n+1}}{C_{n+1} - C_1} - 1 \right) \frac{1}{C_1^{r+1}} \right.$$

$$\left. - \frac{1}{(C_2 - C_1)(C_3 - C_2)\ldots(C_n - C_2)} \cdot \left(\frac{C_{n+1}}{C_{n+1} - C_2} - 1 \right) \frac{1}{C_2^{r+1}} + \&c. \right\}$$

$$= C_1 C_2 \ldots C_n \left\{ \frac{1}{(C_2 - C_1)(C_3 - C_1)\ldots(C_{n+1} - C_1)} \cdot \frac{1}{C_1^{r}} \right.$$

$$\left. - \frac{1}{(C_2 - C_1)(C_3 - C_2)\ldots(C_{n+1} - C_2)} \cdot \frac{1}{C_2^{r}} + \&c. \right\}$$

$$= \frac{1}{C_{n+1}} \cdot F_{r-1}(n+1),$$

i.e. $\Delta F_r(n) = \dfrac{F_{r-1}(n+1)}{C_{n+1}}$, but $\Delta \phi_r(n) = \dfrac{\phi_{r-1}(n+1)}{C_{n+1}}$,

hence $\phi_r(n)$ may be put for $F_r(n)$, and we have

$$\phi_r(n) = C_1 C_2 \ldots C_n \left\{ \frac{1}{(C_2 - C_1)(C_3 - C_1)\ldots(C_n - C_1)} \cdot \frac{1}{C_r^{r+1}} \right.$$

$$\left. - \frac{1}{(C_2 - C_1)(C_3 - C_2)\ldots(C_n - C_2)} \cdot \frac{1}{C_2^{r+1}} + \&c. \right\} \ldots \text{(I)}.$$

2. Required to prove that if

$$\psi_r(n) = \frac{1}{C_1^r} + \frac{1}{C_2^r} + \ldots + \frac{1}{C_n^r},$$

then $\log\{1 + \phi_1(n).x + \phi_2(n).x^2 + \phi_3(n).x^3 + \&c.\}$

$$= \psi_1(n) \cdot \frac{x}{2} + \psi_2(n) \cdot \frac{x^2}{2} + \psi_3(n) \cdot \frac{x^3}{3} + \&c. \ldots\ldots \text{(II)}.$$

This formula may be proved in its successive cases as follows:

By expanding and equating coefficients, we have (as requiring to be proved)

(1) $\psi_1(n) = \phi_1(n)$

which is an identity,

(2) $\psi_2(n) = 2\phi_2(n) - \phi_1^2(n)$,

(3) $\psi_3(n) = 3\phi_3(n) - 3\phi_1(n)\,\phi_2(n) + \phi_1^3(n)$,

(4) $\psi_4(n) = 4\phi_4(n) - 2\phi_2^2(n) - 4\phi_1(n)\,\phi_3(n) + 4\phi_1^2(n)\,\phi_2(n) - \phi_1^4(n)$

and so on.

[1] Let $\qquad f(n) = 2\phi_2(n) - \phi_1^2(n)$,

then $\Delta f(n) = 2\{\phi_2(n+1) - \phi_2(n)\} - \{\phi_1^2(n+1) - \phi_1^2(n)\}$

$$= 2\Delta\phi_2(n) - \{\phi_1(n+1) - \phi_1(n)\}\{\phi_1(n+1) + \phi_1(n)\}$$

$$= \frac{2\phi_1(n+1)}{C_{n+1}} - \frac{1}{C_{n+1}}\left\{2\phi_1(n) + \frac{1}{C_{n+1}}\right\} \text{ by (1)}$$

$$= \frac{2}{C_{n+1}}\{\phi_1(n+1) - \phi_1(n)\} - \frac{1}{C_{n+1}^2} = \frac{2}{C_{n+1}^2} - \frac{1}{C_{n+1}^2} = \frac{1}{C_{n+1}^2} = \Delta\psi_2(n),$$

hence $\psi_2(n) = f(n) = 2\phi_2(n) - \phi_1^2(n)$, no correction in the integration being required, since $\phi_1(n)$, $\phi_2(n)$, $\psi_2(n)$ which signify respectively the sums of n terms of a series necessarily vanish when $n = 0$.

[2] Let $f(n) = 3\phi_3(n) - 3\phi_1(n)\,\phi_2(n) + \phi_1^3(n)$,

then $\Delta f(n) = 3\{\phi_3(n+1) - \phi_3(n)\}$

$$- 3\{\phi_1(n+1)\,\phi_2(n+1) - \phi_1(n)\,\phi_2(n)\} + \phi_1(n+1)^3 - \phi_1^3(n)$$

$$= \frac{3\phi_2(n+1)}{C_{n+1}} - 3\left[\left\{\phi_1(n) + \frac{1}{C_{n+1}}\right\}\left\{\phi_2(n) + \frac{\phi_1(n+1)}{C_{n+1}}\right\} - \phi_1(n)\,\phi_2(n)\right] + \left\{\phi_1(n) + \frac{1}{C_{n+1}}\right\}^3 - \phi_1^3(n)$$

$$= \frac{3\phi_2(n+1)}{C_{n+1}} - 3\left\{\frac{\phi_2(n)}{C_{n+1}} + \frac{\phi_1(n)\,\phi_1(n+1)}{C_{n+1}} + \frac{\phi_1(n+1)}{C_{n+1}^2}\right\} + \frac{3\phi_1^2(n)}{C_{n+1}} + \frac{3\phi_1(n)}{C_{n+1}^2} + \frac{1}{C_{n+1}^3}$$

$$= \frac{3}{C_{n+1}}\{\phi_2(n+1) - \phi_2(n)\} - \frac{3\phi_1(n)}{C_{n+1}}\{\phi_1(n+1) - \phi_1(n)\} - \frac{3}{C_{n+1}^2}\{\phi_1(n+1) - \phi_1(n)\} + \frac{1}{C_{n+1}^3}$$

$$= \frac{3\phi_1(n+1)}{C_{n+1}^2} - \frac{3\phi_1(n)}{C_{n+1}^2} - \frac{3}{C_{n+1}^3} + \frac{1}{C_{n+1}^3} = \frac{1}{C_{n+1}^3} = \Delta\psi_3(n),$$

hence $\psi_3(n) = f(n) = 3\phi_3(n) - 3\phi_1(n)\ \phi_2(n) + \phi_1(n)^3$.

In a similar manner it may be shewn that

$$\psi_4(n) = 4\phi_4(n) - 2\phi_2^{\,2}(n) - 4\phi_1(n)\ \phi_3(n) + 4\phi^2(n)\ \phi_2(n) - \phi_1^{\,4}(n)$$

and so on.*

3. Since C_n is an arbitrary function of n, the formula (I) is susceptible of an indefinite number of applications. The simplest and most important probably of these are the following:

[1] Let $$C_n = an + b,$$

then $$\phi_1(n) = \frac{1}{a+b} + \frac{1}{2a+b} + \ldots + \frac{1}{na+b},$$

and from (I) we have

$$\phi_r(n) = (a+b)(2a+b)\ldots(na+b)\left[\frac{1}{a(2a)(3a)\ldots\{(n-1)a\}}\cdot\frac{1}{(a+b)^{r+1}}\right.$$
$$- \frac{1}{a(a)(2a)\ldots\{(n-2)a\}}\cdot\frac{1}{(2a+b)^{r+1}}$$
$$\left.+ \frac{1}{2a(a)(a)\ldots\{(n-3)a\}}\cdot\frac{1}{(3a+b)^{r+1}} - \&c.\right]$$
$$= \frac{a\Gamma\left(n+1+\frac{b}{a}\right)}{\Gamma n\,\Gamma\left(1+\frac{b}{a}\right)}\left\{\frac{1}{(a+b)^{r+1}} - \frac{n-1}{1}\cdot\frac{1}{(2a+b)^{r+1}}\right.$$
$$\left.+ \frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{(3a+b)^{r+1}} + \&c.\right\};$$

let $r = 1$, then

$$\phi_1(n)\left(= \frac{1}{a+b} + \frac{1}{2a+b} + \ldots + \frac{1}{na+b}\right)$$
$$= \frac{a\Gamma\left(n+1+\frac{b}{a}\right)}{\Gamma n\,\Gamma\left(1+\frac{b}{a}\right)}\left\{\frac{1}{(a+b)^2} - \frac{n-1}{1}\cdot\frac{1}{(2a+b)^2} + \&c.\right\}.$$

Ex. (1) $$a = 1,\ b = 0,$$

then $$\phi_r(n) = \frac{n}{1}\cdot\frac{1}{1^r} - \frac{n(n-1)}{1.2}\cdot\frac{1}{2^r} + \frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{3^r} - \&c.,$$

* A general proof of Formula II. will be subsequently given.

$$(r=1)\ \phi_1(n)\left(=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{n}{1}\cdot\frac{1}{1}-\frac{n(n-1)}{1.2}\cdot\frac{1}{2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{3}-\&c.\ \ldots\ldots(2)$$

a well known formula.

Ex. (2) $\qquad a=2,\ \ b=-1,$

then $\qquad \phi_1(n)=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1},$

and $\phi_r(n)=\frac{2\Gamma(n+\frac{1}{2})}{\Gamma n\,\Gamma\frac{1}{2}}\left\{\frac{1}{1^{r+1}}-\frac{n-1}{1}\cdot\frac{1}{3^{r+1}}+\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{5^{r+1}}-\&c.\right\},$

$$\phi_1(n)\left(=\frac{1}{1}+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)$$

$$=\frac{2\Gamma(n+\frac{1}{2})}{\Gamma n\,\Gamma\frac{1}{2}}\left\{\frac{1}{1^2}-\frac{n-1}{1}\cdot\frac{1}{3^2}+\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{5^2}-\&c.\right\}\ldots(3).$$

Ex. (3) $\qquad a=3,\ b=-2,$

then $\phi_1(n)\left(=\frac{1}{1}+\frac{1}{4}+\frac{1}{7}+\ldots+\frac{1}{3n-2}\right)$

$$=\frac{3\Gamma(n+\frac{1}{3})}{\Gamma n\,\Gamma\frac{1}{3}}\left\{\frac{1}{1^2}-\frac{n-1}{1}\cdot\frac{1}{4^2}+\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{7^2}-\&c.\right\}\ldots(4),$$

if $n=3,\ \frac{1}{1}+\frac{1}{4}+\frac{1}{7}\left(=\frac{39}{28}\right)$

$$=\frac{3\Gamma\frac{10}{3}}{2\Gamma\frac{1}{3}}\left(1-\frac{1}{8}+\frac{1}{49}\right)=\frac{14}{9}\cdot\frac{351}{8\cdot 49}=\frac{39}{28}.$$

[2] Let $C_n=n^2$, then $\phi_1(n)=\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2},$
and from (I) we obtain

$$\phi_r(n)=(1.2.3\ldots n)^2\left\{\frac{1}{(2^2-1^2)(3^2-1^2)\ldots(n^2-1^2)}\cdot\frac{1}{1^{2(r+1)}}\right.$$
$$\left.-\frac{1}{(2^2-1^2)(3^2-2^2)\ldots(n^2-2^2)}\cdot\frac{1}{2^{2(r+1)}}+\&c.\right\},$$

which becomes on reduction

$$\phi_r(n)=2\left\{\frac{n}{n+1}\cdot\frac{1}{1^{2r}}-\frac{n(n-1)}{(n+1)(n+2)}\cdot\frac{1}{2^{2r}}\right.$$
$$\left.+\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\cdot\frac{1}{3^{2r}}-\&c.\right\}.$$

Hence $\phi_1(n)\left(=\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)$

$$=2\left\{\frac{n}{n+1}\cdot\frac{1}{1^2}-\frac{n(n-1)}{(n+1)(n+2)}\cdot\frac{1}{2^2}\right.$$
$$\left.+\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\cdot\frac{1}{3^2}-\text{\&c.}\right\}\ldots\ldots(5).$$

[3] Let $C_n=\frac{1}{n}$, then $\phi_1(n)=1+2+3+\ldots+n=\frac{n(n+1)}{2}$, and from (I)

$$\phi_r(n)=\frac{(-1)^{n-1}}{1.2.3\ldots n}\left\{\frac{1}{\left(\frac{1}{1}-\frac{1}{2}\right)\left(\frac{1}{1}-\frac{1}{3}\right)\ldots\left(\frac{1}{1}-\frac{1}{n}\right)}.1^{r+1}\right.$$
$$\left.-\frac{1}{\left(\frac{1}{1}-\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{3}\right)\ldots\left(\frac{1}{2}-\frac{1}{n}\right)}.2^{r+1}+\text{\&c.}\right\}$$
$$=\frac{(-1)^{n-1}}{\Gamma n}\left\{1^{n+r-1}-\frac{n-1}{1}.2^{n+r-1}+\frac{(n-1)(n-2)}{1.2}.3^{n+r-1}-\text{\&c.}\right\}.$$

Hence $\phi_1(n)\left\{=\frac{n(n+1)}{2}\right\}$

$$=\frac{(-1)^{n-1}}{\Gamma n}\left\{1^n-\frac{n-1}{1}.2^n+\frac{(n-1)(n-2)}{1.2}.3^n-\text{\&c.}\right\}\ldots\ldots(6).$$

Cor. 1. $1^n-\frac{n-1}{1}.2^n+\frac{(n-1)(n-2)}{1.2}.3^n-\text{\&c.}$

$$=\frac{(-1)^{n-1}}{2}.\Gamma(n+2)\ldots\ldots\ldots(7).$$

Cor. 2. Since $\phi_2(n)$ (by its construction)

$$=\Sigma^0_n\left(n.\frac{n^2+n}{2}\right)=\frac{3n^4+10n^3+9n^2+2n}{24};$$

therefore $1^{n+1}-\frac{n-1}{1}.2^{n+1}+\frac{(n-1)(n-2)}{1.2}.3^{n+1}-\text{\&c.}$

$$=(-1)^{n-1}\Gamma n\left(\frac{3n^4+10n^3+9n^2+2n}{24}\right).$$

Hence generally (putting $n+1$ for n)

$$1^{n+r}-\frac{n}{1}.2^{n+r}+\frac{n(n-1)}{1.2}.3^{n+r}-\text{\&c.}=(-1)^n\Gamma(n+1)P_n\ldots(8),$$

where P_n is a function of n of $2r$ dimensions.

[4] Let $$C_n = \frac{1}{n^2},$$

then $\phi_1(n) = 1^2 + 2^2 + \ldots + n^2 = \Sigma_n^0(n^2) = \dfrac{2n^3 + 3n^2 + n}{6}$,

and we have from (I), after reduction,

$$\phi_r(n) = \frac{2(-1)^{n-1}}{\Gamma(n+1)^2}\left\{\frac{n}{n+1}.1^{2(n+r)} - \frac{n(n-1)}{(n+1)(n+2)}.2^{2(n+r)}\right.$$
$$\left. + \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}.3^{2(n+r)} - \&c.\right\},$$

$$\phi_1(n)\left(= \frac{2n^3 + 3n^2 + n}{6}\right)$$
$$= \frac{2(-1)^{n-1}}{\Gamma(n+1)^2}\left\{\frac{n}{n+1}.1^{2(n+1)} - \frac{n(n-1)}{(n+1)(n+2)}.2^{2(n+1)} + \&c.\right\}\ldots(9).$$

Hence $$\frac{n}{n+1}.1^{2(n+1)} - \frac{n(n-1)}{(n+1)(n+2)}.2^{2(n+1)}$$
$$+ \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}.3^{2(n+1)} - \&c. = (-1)^{n-1}.\Gamma(n+1)^2.\frac{2n^3+3n^2+n}{12},$$

also

$$\phi_2(n) = \Sigma^0_n\,\frac{n^2(2n^3+3n^2+n)}{6} = \frac{20n^6+96n^5+155n^4+90n^3+5n^2-6n}{360},$$

which therefore

$$= \frac{2(-1)^{n-1}}{\Gamma(n+1)^2}\left\{\frac{n}{n+1}.1^{2(n+2)} - \frac{n(n-1)}{(n+1)(n+2)}.2^{2(n+2)} + \&c.\right\}\ldots(10),$$

and so on.

Ex. $(n=2)$, then $21 = \dfrac{2(-1)}{4}\left(\dfrac{2}{3} - \dfrac{1}{6}.2^8\right) = -\left(\dfrac{1}{3} - \dfrac{64}{3}\right) = 21.$

[5] Let $$C_n = n(n+1),$$

then $$\phi_1(n) = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1},$$

and we obtain from (I), after reduction,

$$\phi_r(n) = \frac{n}{n+2}\cdot\frac{3}{(1.2)^r} - \frac{n(n-1)}{(n+2)(n+3)}\cdot\frac{5}{(2.3)^r}$$
$$+ \frac{n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\cdot\frac{7}{(3.4)^r} - \&c.,$$

let $$r = 1,$$

then $$\phi_1(n)\left(=\frac{n}{n+1}\right) = \frac{n}{n+2}\cdot\frac{3}{1.2} - \frac{n(n-1)}{(n+2)(n+3)}\cdot\frac{5}{2.3} + \&c.,$$

and $$\frac{1}{n+1} = \frac{1}{n+2}\cdot\frac{3}{1.2} - \frac{n-1}{(n+2)(n+3)}\cdot\frac{5}{2.3}$$
$$+ \frac{(n-1)(n-2)}{(n+2)(n+3)(n+4)}\cdot\frac{7}{3.4} - \&c. \;\ldots\ldots(11).$$

[6] Let $$C_n = (2n-1)^2,$$

then $$\phi_1(n) = \frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{(2n-1)^2},$$

and from (I) we have

$$\phi_r(n) = n\left\{\frac{2\,\Gamma(n+\frac{1}{2})}{\Gamma\frac{1}{2}\,\Gamma(n+1)}\right\}^2\cdot\left\{\frac{1}{1^{2r+1}} - \frac{n-1}{n+1}\cdot\frac{1}{3^{2r+1}}\right.$$
$$\left. + \frac{(n-1)(n-2)}{(n+1)(n+2)}\cdot\frac{1}{5^{2r+1}} - \&c.\right\},$$

hence $$\phi_1(n)\left\{=\frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{(2n-1)^2}\right\}$$

$$= n\left\{\frac{2\,\Gamma(n+\frac{1}{2})}{\Gamma\frac{1}{2}\,\Gamma(n+1)}\right\}^2\cdot\left\{\frac{1}{1^3} - \frac{n-1}{n+1}\cdot\frac{1}{3^3} + \frac{(n-1)(n-2)}{(n+1)(n+2)}\cdot\frac{1}{5^3} + \&c.\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Ex. $$(n = 3),$$

then $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\left(=\frac{259}{225}\right) = \frac{75}{64}\left(1 - \frac{1}{2.27} + \frac{1}{2.625}\right),$$

which is the case.

[7] Let $$C_n = (2n-1)\,2n,$$

then $$\phi_1(n) = \frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2n-1)\,2n},$$

then from (I) we obtain

$$\phi_r(n) = \frac{2n}{2n+1}\cdot\frac{3}{(1.2)^{r+1}} - \frac{3}{2}\cdot\frac{2n(2n-2)}{(2n+1)(2n+3)}\cdot\frac{7}{(3.4)^{r+1}}$$
$$+ \frac{3.5}{2.4}\cdot\frac{2n(2n-2)(2n-4)}{(2n+1)(2n+3)(2n+5)}\cdot\frac{11}{(5.6)^{r+1}} - \&c.$$

Hence $\phi_1(n)\left\{=\frac{1}{1.2}+\frac{1}{3.4}+\ldots+\frac{1}{(2n-1)\,2n}\right\}$

$$=\frac{2n}{(2n+1)}\cdot\frac{3}{(1.2)}-\frac{3}{2}\cdot\frac{2n\,(2n-2)}{(2n+1)\,(2n+3)}\cdot\frac{7}{(3.4)^2}+\&c.\ldots(13).$$

Ex. $(n=3)$, then $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}\left(=\frac{37}{60}\right)$

$$=\frac{6}{7}\cdot\frac{3}{2^2}-\frac{3}{2}\cdot\frac{6.4}{7.9}\cdot\frac{7}{12^2}+\frac{3.5}{2.4}\cdot\frac{6.4.2}{7.9.11}\cdot\frac{11}{30^2},$$

which is the case.

[8] Let $C_n=n^3$, then $\phi_1(n)=\frac{1}{1^3}+\frac{1}{2^3}+\ldots+\frac{1}{n^3}$,

and we obtain from (I), after considerable reduction,

$$\phi_r(n)=3\Gamma(n+1)^2\left\{\frac{n}{1}\cdot\frac{\Gamma(1-\rho)\,\Gamma(1-\rho^2)}{\Gamma(n+1-\rho)\,\Gamma(n+1-\rho^2)}\cdot\frac{1}{1^{3r}}\right.$$
$$\left.-\frac{n\,(n-1)}{1.2}\,\frac{\Gamma(1-2\rho)\,(1-2\rho^2)}{\Gamma(n+1-2\rho)\,\Gamma(n+1-2\rho^2)}\cdot\frac{1}{2^{3r}}+\&c.\right\},$$

where $\rho^3=1$, hence putting $r=1$,

$$\phi_1(n)\left(=\frac{1}{1^3}+\frac{1}{2^3}+\ldots+\frac{1}{n^3}\right)=3\Gamma(n+1)^2\left\{\frac{n}{1}\,\frac{\Gamma(1-\rho)\,\Gamma(1-\rho^2)}{\Gamma(n+1-\rho)\,\Gamma(n+1-\rho^2)}\cdot\frac{1}{1^3}\right.$$
$$-\frac{n\,(n-1)}{1.2}\,\frac{\Gamma(1-2\rho)\,\Gamma(1-2\rho^2)}{\Gamma(n+1-2\rho)\,\Gamma(n+1-2\rho^2)}\cdot\frac{1}{2^3}$$
$$\left.+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{\Gamma(1-3\rho)\,\Gamma(1-3\rho^2)}{\Gamma(n+1-3\rho)\,\Gamma(n+1-3\rho^2)}\cdot\frac{1}{3^3}-\&c.\right\}\ldots(14).$$

Ex. $(n=2)$,

then $\frac{1}{1^3}+\frac{1}{2^3}\left(=\frac{9}{8}\right)=3\,(1.2)^2\left\{\frac{2}{1}\cdot\frac{\Gamma(1-\rho)\,\Gamma(1-\rho^2)}{\Gamma(3-\rho)\,\Gamma(3-\rho^2)}\cdot\frac{1}{1^3}\right.$

$$\left.-\frac{2.1}{1.2}\,\frac{\Gamma(1-2\rho)\,\Gamma(1-2\rho^2)}{\Gamma(3-2\rho)\,\Gamma(3-2\rho^2)}\cdot\frac{1}{2^3}\right\}$$

$$=12\left\{\frac{2}{1}\cdot\frac{1}{(1-\rho)\,(1-\rho^2)\,(2-\rho)\,(2-\rho^2)}\cdot\frac{1}{1^3}\right.$$
$$\left.-\frac{1}{(1-2\rho)\,(1-2\rho^2)\,(2-2\rho)\,(2-2\rho^2)}\cdot\frac{1}{2^3}\right\}$$

$$=12\left\{\frac{2}{1}\cdot\frac{1}{7.3}\cdot\frac{1}{1^3}-\frac{1}{12.7}\cdot\frac{1}{2^3}\right\}=\frac{24}{21}-\frac{1}{56}=\frac{9}{8}.$$

[9] Let $C_n = n^4$, then $\phi_1(n) = \frac{1}{1^4} + \frac{1}{2^4} + \ldots + \frac{1}{n^4}$, and from (I)

$$\phi^r(n) = 4\Gamma(n+1)^2 \left\{ \frac{n}{n+1} \cdot \frac{\Gamma(1+\rho)\,\Gamma(1-\rho)}{\Gamma(n+1+\rho)\,\Gamma(n+1-\rho)} \cdot \frac{1}{1^{4r}} \right.$$

$$\left. - \frac{n(n-1)}{(n+1)(n+2)} \cdot \frac{\Gamma(1+2\rho)\,\Gamma(1-2\rho)}{\Gamma(n+1+2\rho)\,\Gamma(n+1-2\rho)} \cdot \frac{1}{2^{4r}} + \&c. \right\},$$

where $\rho^4 = 1$, and therefore $\rho = \sqrt{(-1)}$. Hence $(r=1)$

$$\phi_1(n) \left(= \frac{1}{1^4} + \frac{1}{2^4} + \ldots + \frac{1}{n^4} \right)$$

$$= 4\Gamma(n+1)^2 \left\{ \frac{n}{n+1} \cdot \frac{\Gamma(1+\rho)\,\Gamma(1-\rho)}{\Gamma(n+1+\rho)\,\Gamma(n+1-\rho)} \cdot \frac{1}{1^4} \right.$$

$$\left. - \frac{n(n-1)}{(n+1)(n+2)} \cdot \frac{\Gamma(1+2\rho)\,\Gamma(1-2\rho)}{\Gamma(n+1+2\rho)\,\Gamma(n+1-2\rho)} \cdot \frac{1}{2^4} + \&c. \right\} \ldots (15).$$

Ex. $(n=2)$, then $\frac{1}{1^4} + \frac{1}{2^4} \left(= \frac{17}{16} \right)$

$$= 4.4 \left\{ \frac{2}{3} \cdot \frac{\Gamma(1+\rho)\,\Gamma(1-\rho)}{\Gamma(3+\rho)\,\Gamma(3-\rho)} \cdot \frac{1}{1^4} \right.$$

$$\left. - \frac{2.1}{3.4} \cdot \frac{\Gamma(1+2\rho)\,\Gamma(1-2\rho)}{\Gamma(3+2\rho)\,\Gamma(3-2\rho)} \cdot \frac{1}{2^4} + \&c. \right\}$$

$$= 16 \left\{ \frac{2}{3} \cdot \frac{1}{(1+\rho)(1-\rho)(2+\rho)(2-\rho)} \cdot \frac{1}{1^4} \right.$$

$$\left. - \frac{1}{6} \cdot \frac{1}{(1+2\rho)(1-2\rho)(2+2\rho)(2-2\rho)} \cdot \frac{1}{2^4} + \&c. \right\}$$

$$= 16 \left(\frac{1}{15} \cdot \frac{1}{1^4} - \frac{1}{6} \cdot \frac{1}{40} \cdot \frac{1}{2^4} \right) = \frac{16}{15} - \frac{1}{240} = \frac{255}{240} = \frac{17}{16}.$$

4. The preceding formulæ all belong to a peculiar class of equations which possess a restricted equality, holding good only within certain limits. The proof above given depends entirely on the supposition that the quantity n which is involved is a positive integer, and it is doubtful whether the resources of common Algebra can extend the proof beyond this limitation. When n is fractional or negative, these formulæ become transcendents, and for their evaluation

it is necessary to ascertain within what limits, as regards the value of n, they hold good. The investigation of those limits does not appear to be altogether easy. We shall be able, however, by use of representative notation, to determine the true limits and range of application of the most important of these formulæ and thus to arrive at some very remarkable results.

(*To be continued.*)

DETERMINATION OF THE TRILINEAR EQUATION TO THE AXES OF A CONIC SECTION.

By WILLIAM ALLEN WHITWORTH, B.A., Scholar of St. John's College.

THE following method of obtaining this equation is perhaps more direct than the very elegant, though somewhat laborious method given by Mr. Hensley in pp. 273–275 of the present volume.

Let $\lambda, \mu, \nu, \lambda', \mu', \nu'$, be the direction sines of the two tangents that can be drawn from a point $(\alpha', \beta', \gamma')$ to the conic whose equation is

$$f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0;$$

then the length of the first tangent is given by the equation

$$f(\alpha', \beta', \gamma') + \rho\left(\lambda\frac{df}{d\alpha'} + \mu\frac{df}{d\beta'} + \nu\frac{df}{d\gamma'}\right) + \rho^2 f(\lambda, \mu, \nu) = 0,$$

and the roots of this equation must be equal, therefore

$$\left(\lambda\frac{df}{d\alpha'} + \mu\frac{df}{d\beta'} + \nu\frac{df}{d\gamma'}\right)^2 = 4f(\alpha', \beta', \gamma')f(\lambda, \mu, \nu).$$

Similarly

$$\left(\lambda'\frac{df}{d\alpha'} + \mu'\frac{df}{d\beta'} + \nu'\frac{df}{d\gamma'}\right)^2 = 4f(\alpha', \beta', \gamma')f(\lambda', \mu', \nu').$$

Now suppose $(\alpha', \beta', \gamma')$ be any point on either of the axes of the conic: then the two tangents are equal, and therefore

$$f(\lambda, \mu, \nu) = f(\lambda', \mu', \nu') \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

consequently

$$\left(\lambda\frac{df}{d\alpha} + \mu\frac{df}{d\beta} + \nu\frac{df}{d\gamma}\right)^2 = \left(\lambda'\frac{df}{d\alpha'} + \mu'\frac{df}{d\beta'} + \nu'\frac{df}{d\gamma'}\right)^2 \ldots(2).$$

Also, by the identical relations which always exist among the direction sines of any straight line,

$$\mu^2 + \nu^2 + 2\mu\nu\cos A = \sin^2 A = \mu'^2 + \nu'^2 + 2\mu'\nu'\cos A \ldots(3),$$

$$\nu^2 + \lambda^2 + 2\nu\lambda\cos B = \sin^2 B = \nu'^2 + \lambda'^2 + 2\nu'\lambda'\cos B \ldots(4),$$

$$\lambda^2 + \mu^2 + 2\lambda\mu\cos C = \sin^2 C = \lambda'^2 + \mu'^2 + 2\lambda'\mu'\cos C \ldots(5),$$

$$(\lambda\sin A + \mu\sin B + \nu\sin C)^2 = (\lambda'\sin A + \mu'\sin B + \nu'\sin C)^2 \ldots\ldots\ldots\ldots\ldots(6).$$

Eliminating the six quantities

$$\lambda^2 - \lambda'^2,\ \mu^2 - \mu'^2,\ \nu^2 - \nu'^2,\ \mu\nu - \mu'\nu',\ \nu\lambda - \nu'\lambda',\ \lambda\mu - \lambda'\mu',$$

from these equations, and suppressing the accents on the coordinates, we obtain

$$\begin{vmatrix} \left(\frac{df}{d\alpha}\right)^2, & \left(\frac{df}{d\beta}\right)^2, & \left(\frac{df}{d\gamma}\right)^2, & \frac{df}{d\beta}\frac{df}{d\gamma}, & \frac{df}{d\gamma}\frac{df}{d\alpha}, & \frac{df}{d\alpha}\frac{df}{d\beta}, \\ \sin^2 A, & \sin^2 B, & \sin^2 C, & \sin B\sin C, & \sin C\sin A, & \sin A\sin B, \\ u, & v, & w, & u', & v', & w', \\ 0, & 1, & 1, & \cos A, & 0, & 0, \\ 1, & 0, & 1, & 0, & \cos B, & 0, \\ 1, & 1, & 0, & 0, & 0, & \cos C, \end{vmatrix} = 0,$$

a relation of the second order between the coordinates of any point on either axis, and therefore the equation to the axes.

Runcorn, Cheshire,
June 20th, 1862.

ON THE THEORY OF THE TRANSCENDENTAL SOLUTION OF ALGEBRAIC EQUATIONS.

By the Rev. ROBERT HARLEY, F.R.A.S., Corresponding Member of the Literary and Philosophical Society of Manchester.

1. FINITE algebraic solutions of equations of the first four degrees have long been known. Bombelli solved cubic equations falling within the irreducible case by circular functions, and it seems that by such functions the roots of all cubics may be expressed.* Demoivre extended Bombelli's method of solution to certain forms of quintics, and it may be readily applied to quadratics. Spence has given a similar (trigonometric) solution for the biquadratic. And the recent researches of M. M. Hermite and Kronecher show that a solution somewhat analogous to that by circular functions exists for equations of the fifth degree. M. Hermite availing himself of Mr. Jerrard's transformation of those equations to trinomial forms involving only one parameter, has succeeded in expressing the roots of the quintic in terms of elliptic functions. M. Kronecher has done the same thing by processes that do not require any preliminary modification of the coefficients of the general quintic.† So that we have solutions, algebraic, trigonometric, or transcendental, of all algebraic equations of a degree lower than the sixth. But the methods of arriving at these solutions are various and independent; and although an ultimate uniformity runs through the algebraic solutions, no apparent connection subsists between them and the others, which are moreover indirect. It seems desirable therefore to discover if possible some not only direct but uniform process, and so to bind together, *unico vinculo*, these isolated methods of solution. The more comprehensive the process the greater certainly its advantages, and strong or even conclusive as may be the presumption against the possibility of solving algebraic equations in general by radicals, there is, as yet, none whatever against the possibility of solving them by integrals.

2. About a year and a-half ago Mr. Cockle communicated to me by letter some of his researches on the theory of trans-

* See Todhunter's *Theory of Equations*, pp. 100, 101.

† See pp. 25-27 of M. Hermite's Essay entitled *Sur la Théorie des Equationes Modulaires et la Resolution de l'Equation du Cinquième Degré.*

cendental roots. I was greatly interested in his treatment of the subject, because it seemed to indicate a principle of solution applicable to every algebraic equation which can be reduced to a form involving only one parameter; but I was too much occupied at the time with some other researches to do more than read what my friend had written and make memoranda of a few developments which in reading occurred to me.

3. It may be convenient to mention here that Mr. Cockle has published* some investigations on this subject, and also that he has found that no restriction as to the number of arbitrary coefficients is necessary, but that the same process is applicable to any equation of any degree without any preliminary modification of its coefficients and without having recourse to more than one independent variable. I propose in the present paper to avail myself freely of what Mr. Cockle has done and to present in a systematic form some of the applications of which his method is susceptible.

4. We know that any general equation of a degree not higher than the fifth can be transformed into another, say $fy=0$, whose coefficients are all functions of a single parameter x, and therefore that the roots of such an equation are functions of x only. It will be found greatly to facilitate the calculations and to simplify the exposition of the method, if we deal with the equation under this reduced form. How the processes may be extended so as to embrace any general equation, whether it be capable of the foregoing transformation or not, is an inquiry which we shall postpone to future consideration.

5. Since the equation $fy=0$ must be satisfied identically by any one of its n roots, it follows that the successive derived functions

$$\frac{dfy}{dx}, \quad \frac{d^2fy}{dx^2}, \ \ldots \ \frac{d^{n-1}fy}{dx^{n-1}},$$

* See his "Sketch of a Theory of Transcendental Roots" in the *Philosophical Magazine*, for August, 1860, and his "Note on Transcendental Roots" in the ensuing November Number of that Journal; see also the concluding article of his "Note on the Remarks of Mr. Jerrard" in the Number for February, 1861, and his paper "On Transcendental and Algebraic Solution" in the ensuing May Number. A paper supplementary to the last will also be found in the Number for February of the present year.

must vanish identically. By actual differentiation we shall evidently obtain a result of the form

$$\frac{dfy}{dx} = R_1 y . \frac{dy}{dx} + R_2 y = 0,$$

and, since R_1 and R_2 are rational functions, we have, by a known process,

$$\frac{dy}{dx} = -\frac{R_2 y}{R_1 y} = Ry,$$

or

$$\frac{dy}{dx} = X_1' y^{n-1} + X_2' y^{n-2} \ldots + X'_{n-1},$$

where X' is a function of x. Differentiating again, substituting for $\frac{dy}{dx}$ in the result the value above determined, and performing necessary reductions, we find

$$\frac{d^2y}{dx^2} = X_1'' y^{n-1} + X_2'' y^{n-2} \ldots + X''_{n-1}.$$

Repeating this process we are conducted to the system

$$\begin{array}{llll}
\frac{dy}{dx} = X_1' y^{n-1} & + X_2' y^{n-2} & \ldots + X'_{n-1}, \\
\frac{d^2y}{dx^2} = X_1'' y^{n-1} & + X_2'' y^{n-2} & \ldots + X''_{n-1}, \\
\vdots & \vdots & \vdots \\
\frac{d^{n-1}y}{dx^{n-1}} = X_1^{(n-1)} y^{n-1} & + X_2^{(n-1)} y^{n-2} & \ldots + X^{(n-1)}_{n-1},
\end{array}$$

which, by means of the $n-2$ indeterminate multipliers $\mu_1, \mu_2, \ldots \mu_{n-2}$, so assigned as to cause all powers of y higher than the first to disappear, may be made to yield

$$\frac{d^{n-1}y}{dx^{n-1}} + \mu_1 \frac{d^{n-2}y}{dx^{n-2}} \ldots + \mu_{n-2} \frac{dy}{dx} + X_0 y + X_1 = 0,$$

a linear differential equation, μ and X being known functions of x.

6. It is proposed to call this equation the "differential resolvent." There are other methods of arriving at it besides that above given; and some of these, in dealing with special cases, I shall notice hereafter. The solution of the differential resolvent involves of course the solution of the equation in y. For, any of the n roots $y_1, y_2, \ldots y_n$ of the

equation in y, and therefore also any of the constituents of those roots must satisfy the differential resolvent. For the sake of shortness let us represent the resolvent whose form is determined in the last article by $\phi y = 0$; then referring to that form, it is easy to see that

$$\phi Y = a_0 X_0 + \{1 - (a_1 + a_2 \ldots + a_n)\} X_1,^*$$

where Y is a linear function of y and of the form

$$a_0 + a_1 y_1 + a_2 y_2 \ldots + a_n y_n,$$

and $a_0, a_1, a_2, \ldots a_n$ are arbitrary constants.

Now we know by Lagrange's theory that the several constituents of y are of the form

$$\frac{1}{n} \Sigma y + \omega y_1 + \omega^2 y_2 \ldots + \omega^n y_n,$$

or what is the same thing

$$\left(\frac{1}{n} + \omega\right) y_1 + \left(\frac{1}{n} + \omega^2\right) y_2 \ldots + \left(\frac{1}{n} + \omega^n\right) y_n,$$

ω denoting as usual an unreal n^{th} root of unity. And if we put

$$a_0 = 0, \;\; a_1 = \frac{1}{n} + \omega, \;\;\; a_2 = \frac{1}{n} + \omega^2, \; \ldots a_n = \frac{1}{n} + \omega^n,$$

we shall have

$$Y = \left(\frac{1}{n} + \omega\right) y_1 + \left(\frac{1}{n} + \omega^2\right) y_2 \ldots + \left(\frac{1}{n} + \omega^n\right) y_n,$$

and $$\phi Y = 0.$$

Whence it appears that each of the constituents of y will satisfy the differential resolvent $\phi y = 0$; and in fact these constituents are just so many particular integrals of that equation. It follows that every particular integral is a linear function of the constituents, for otherwise there would be more than $n - 1$ independent integrals which is impossible seeing that the resolvent equation is only of the $(n-1)^{\text{th}}$ order. Hence the solution of the differential resolvent, that is, its complete integration, so as to evolve y, or the several constituents of y, in terms of x, will give the required solu-

* When Y and ϕy are homogeneous with respect to y, that is to say, when $a_0 = 0$, and $X_1 = 0$, then ϕY vanishes identically; hence the following theorem: Any homogeneous linear function of the particular integrals of a homogeneous linear differential equation is an integral of that equation.

tion (algebraic, trigonometric, or transcendental) of the equation in y; for, generally, the determination of $n-1$ particular integrals of the differential resolvent will enable us to determine the constituents in question. And here it may be observed that, had we been content with *a posteriori* results, throughout the whole process, no assumption need have been made as to the number of roots or solutions of the given equation; the number of particular integrals of the differential resolvent and of values of certain arbitrary constants would have determined the number of solutions.

7. The method of solution just explained is a modification of that which Mr. Cockle originally proposed, and which in this article I shall briefly indicate.

If we eliminate y between $fy=0$ and $\frac{dfy}{dx}=0$ ($fy=0$ being a trinomial equation of the form given in the next article), there will result an equation of the n^{th} degree in $\frac{dy}{dx}$. Assume that the latter can be transformed by Mr. Jerrard's process or otherwise into another $fy'=0$ of the same form as the given equation, its parameter x' being a known function of x, say $\mathrm{f}x$, and y' being determinable as a function of y, say Fy. Then if we write

$$y=\phi x,$$

no assumption being made as to ϕ, except its fundamental property of satisfying the condition

$$f\phi x=0$$

identically, we shall have

$$y'=\phi x',$$

and therefore

$$F\phi x=\phi \mathrm{f}x,$$

from which to seek the form of ϕ. Concurring with Mr. Cockle in his adoption of the other process I have still thought it desirable to exemplify this also. And accordingly illustrations of it will be found further on.

8. It will now be convenient to select one of the forms involving a single parameter to which it is known that equations of the first five degrees can be reduced. Perhaps there is none more simple or that affords readier verifications than one which Mr. Cockle has chosen, viz.

$$y^n-ny+(n-1)x=0=fy.$$

If in this equation we assign to x certain numerical values, we may easily deduce the corresponding values of y. Thus, when we make $x=0$, we find that the values of y are zero and the $(n-1)^{\text{th}}$ roots of n; and when $x=1$, there are two roots each equal to unity, and the others are given by the equation

$$y^{n-2}+2y^{n-3}+3y^{n-4}\ldots+(n-2)\,y+(n-1)=0.$$

These conditions will be found to give not only a good test of results, but also an easy method of determining the arbitrary constants.

9. Performing the operation $\dfrac{d}{dx}$ on the canonical form

$$fy=y^n-ny+(n-1)\,x,$$

and equating the result with zero, we obtain

$$\frac{dy}{dx}=-\frac{n-1}{n}\cdot\frac{1}{y^{n-1}-1};$$

or since $$y^{n-1}=(n-1)\cdot\frac{y-x}{y}+1,$$

therefore $$\frac{dy}{dx}=-\frac{1}{n}\cdot\frac{y}{y-x}.$$

But $$y\,(y^{n-1}-x^{n-1})-(n-1)\,(y-x)=(1-x^{n-1})\,y,$$

so that $$\frac{y}{y-x}=\frac{1}{1-x^{n-1}}\cdot\left\{\frac{y^{n-1}-x^{n-1}}{y-x}\cdot y-(n-1)\right\},$$

and therefore, since $y^{n-1}-x^{n-1}$ is exactly divisible by $y-x$, the first differential coefficient $\dfrac{dy}{dx}$, or $-\dfrac{1}{n}\cdot\dfrac{y}{y-x}$, is equal to

$$-\frac{1}{n}\cdot\frac{1}{1-x^{n-1}}\cdot\{y^{n-1}+xy^{n-2}+x^2y^{n-3}\ldots+x^{n-2}y-(n-1)\},$$

a rational and integral function of y.

10. It will be noticed that the method by which in the last article the value of the first differential coefficient $\dfrac{dy}{dx}$ is rendered integral differs from the method in common use. It may be worth while however to point out that the same result is given by the ordinary process. We have, in fact,

$$-\frac{y}{y-x}=\frac{y\,(x-y_2)\,(x-y_3)\ldots(x-y_n)}{(x-y)\,(x-y_2)\,(x-y_3)\ldots(x-y_n)};$$

where the denominator of the dexter is obviously what the quantic

$$fy = y^n - ny + (n-1)x$$

becomes when in place of y we write x, viz.

$$fx = x(x^{n-1} - 1);$$

and the numerator is equal to

$$x^{n-1} - x^{n-2}y\dot{\Sigma}y_2 + x^{n-3}y\dot{\Sigma}y_2y_3 \ldots + (-1)^n xy\dot{\Sigma}y_2y_3\ldots y_{n-1}$$
$$- (-1)^n yy_2y_3\ldots y_n,$$

in which expression $\dot{\Sigma}$ is the ordinary Σ symbol applied to the $n-1$ roots $y_2, y_3, \ldots y_n$. But

$$\begin{aligned}
y\dot{\Sigma}y_2 &= y\Sigma y &&- y^2 &&= -y^2,\\
y\dot{\Sigma}y_2y_3 &= y\Sigma yy_2 &&- y^2\dot{\Sigma}y_2 &&= y^3,\\
&\vdots && \vdots && \vdots\\
y\dot{\Sigma}y_2y_3\ldots y_{n-1} &= y\Sigma yy_2\ldots y_{n-1} &&- y^2\dot{\Sigma}y_2y_3\ldots y_{n-2} &&= (-1)^n y^{n-1},\\
yy_2y_3\ldots y_n &= (-1)^n(n-1)x.
\end{aligned}$$

Hence the expression for the numerator becomes

$$x^{n-1}y + x^{n-2}y^2 + x^{n-3}y^3 \ldots + xy^{n-1} - (n-1)x;$$

and therefore $\dfrac{dy}{dx}$, or $-\dfrac{1}{n}\cdot\dfrac{y}{y-x}$, is equal to

$$-\frac{1}{n}\cdot\frac{1}{1-x^{n-1}}\cdot\{x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 \ldots + y^{n-1} - (n-1)\},$$

which coincides, except in the order of terms, with the result obtained in the last article.

11. It will be found interesting as well as useful to notice the following relations, viz.

$$\frac{dy}{dx} = -\frac{n-1}{n}\cdot\frac{1}{y^{n-1}-1};$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{(n-1)^2}{n}\cdot\frac{y^{n-2}}{(y^{n-1}-1)^2}\cdot\frac{dy}{dx}\\
&= -\frac{(n-1)^3}{n}\cdot\frac{y^{n-2}}{(y^{n-1}-1)^3}\\
&= ny^{n-2}\left(\frac{dy}{dx}\right)^3;
\end{aligned}$$

$$\frac{d^3y}{dx^3} = -\frac{(n-1)^3}{n^2} \cdot \frac{(n-2)(y^{n-1}-1)\,y^{n-3} - 3(n-1)\,y^{2n-4}}{(y^{n-1}-1)^4} \cdot \frac{dy}{dx}$$

$$= -\frac{(n-1)^4}{n^3} \cdot \frac{y^{n-3}}{(y^{n-1}-1)^5} \cdot \{(2n-1)\,y^{n-1} + (n-2)\}$$

$$= \frac{n^2}{n-1} \cdot y^{n-3}\,\{(2n-1)\,y^{n-1} + (n-2)\} \left\{\frac{dy}{dx}\right\}^5;$$

$$\frac{d^4y}{dx^4} = -\frac{(n-1)^4}{n^3} \cdot \frac{y^{n-4}}{(y^{n-1}-1)^6} \cdot [(y^{n-1}-1)\,\{(2n-1)(2n-4)\,y^{n-1}$$

$$+ (n-2)(n-3)\} - 5(n-1)\,y^{n-1}\{(2n-1)\,y^{n-1} + (n-2)\}] \left\{\frac{dy}{dx}\right\}$$

$$= -\frac{(n-1)^5}{n^4} \cdot \frac{y^{n-4}}{(y^{n-1}-1)^7} \cdot \{(2n-1)(3n-1)\,y^{2(n-1)}$$

$$+ 4(n-2)(2n-1)\,y^{n-1} + (n-2)(n-3)\}$$

$$= \frac{n^3}{(n-1)^2}\,y^{n-4}\,\{(2n-1)(3n-1)\,y^{2(n-1)} + 4(n-2)(2n-1)\,y^{n-1}$$

$$+ (n-2)(n-3)\} \left\{\frac{dy}{dx}\right\}^7.$$

If in place of the foregoing value of the first differential

$$-\frac{n-1}{n} \cdot \frac{1}{y^{n-1}-1}$$

we had dealt with its equivalent (Art. 9)

$$-\frac{1}{n} \cdot \frac{y}{y-x},$$

we should of course have been led to the same final results. The process in the latter case, though somewhat longer, enables us to evolve certain forms which will be found of value, and I shall here indicate its leading steps

$$\frac{d^2y}{dx^2} = -\frac{1}{n} \cdot \frac{-x\frac{dy}{dx} + y}{(y-x)^2}$$

$$= -\frac{1}{n^2} \cdot \frac{y\,\{ny - (n-1)\,x\}}{(y-x)^3}$$

$$= -\frac{1}{n^2} \cdot \frac{y^{n+1}}{(y-x)^3}$$

$$= ny^{n-2}\left(\frac{dy}{dx}\right)^3;$$

$$\frac{d^3y}{dx^3} = -\frac{y^n}{n^2} \cdot \frac{\{(n+1)(y-x) - 3y\}\dfrac{dy}{dx} + 3y}{(y-x)^4}$$

$$= -\frac{y^{n+1}}{n^3} \cdot \frac{2(n+1)y - (2n-1)x}{(y-x)^5}$$

$$= n^2 y^{n-4} \{2(n+1)y - (2n-1)x\} \left\{\frac{dy}{dx}\right\}^5;$$

or, writing

$$\frac{y(y^{n-1} - n)}{n-1}$$

in place of x, and simplifying, we have, as before,

$$\frac{d^3y}{dx^3} = \frac{n^2}{n-1} y^{n-3} \{(2n-1)y^{n-1} + (n-1)\} \left\{\frac{dy}{dx}\right\}^5.$$

In like manner we find

$$\frac{d^4y}{dx^4} = n^3 y^{n-6} \{3(n+2)(2n+1)y^2 - 2(2n-1)(3n+4)xy$$
$$+ (2n-1)(3n-1)x^2\} \left\{\frac{dy}{dx}\right\}^7;$$

which, on substituting for x its value in terms of y and reducing, gives

$$\frac{d^4y}{dx^4} = \frac{n^3}{(n-1)^2} y^{n-4} \{(2n-1)(3n-1)y^{2(n-1)} + 4(n-2)(2n-1)y^{n-1}$$
$$+ (n-2)(n-3)\} \left\{\frac{dy}{dx}\right\}^7.$$

Similar relations exist for the higher differential coefficients, but it is not necessary to develope them here.

12. Let us now proceed to form the several differential resolvents. That for the quadratic equation

$$y^2 - 2y + x = 0$$

may be deduced immediately from the formula at the foot of article 9 by making $n = 2$. We have in effect

$$\frac{dy}{dx} = -\frac{1}{2} \cdot \frac{1}{1-x}(y-1);$$

and therefore

$$2(1-x)\frac{dy}{dx} + y - 1 = 0.$$

13. For the cubic equation

$$y^3 - 3y + 2x = 0$$

we have by the same formula (making $n = 3$)

$$\frac{dy}{dx} = -\frac{1}{3} \cdot \frac{1}{1 - x^2} \cdot (y^2 + xy - 2);$$

and therefore

$$3(1 - x^2)\frac{dy}{dx} = -(y^2 + xy - 2).$$

Differentiating and transposing, we have

$$3(1 - x^2)\frac{d^2y}{dx^2} = -(2y - 5x)\frac{dy}{dx} - y;$$

or, multiplying into $3(1 - x^2)$ and reducing by means of the foregoing value of $\frac{dy}{dx}$ and the given cubic equation, we have

$$3^2(1 - x^2)^2\frac{d^2y}{dx^2} = -3xy^2 - (1 + 2x^2)\,y + 6x.$$

Combining this equation with the corresponding one in $\frac{dy}{dx}$, and introducing the indeterminate multiplier μ, there results

$$\begin{aligned} 3^2(1 - x^2)^2\frac{d^2y}{dx^2} + 3\mu(1 - x^2)\frac{dy}{dx} = &-(\mu + 3x)\,y^2 \\ &-(1 + \mu x + 2x^2)\,y \\ &+ 2(\mu + 3x); \end{aligned}$$

and assuming, in order to make y^2 vanish,

$$\mu + 3x = 0, \text{ or } \mu = -3x,$$

we have

$$3^2(1 - x^2)^2\frac{d^2y}{dx^2} - 3^2x(1 - x^2)\frac{dy}{dx} = -(1 - x^2)\,y;$$

or, transposing and dividing by $1 - x$,

$$3^2(1 - x^2)\frac{d^2y}{dx^2} - 3^2x\frac{dy}{dx} + y = 0,$$

the differential resolvent for the cubic.

14. The last result may be verified by an independent calculation. I give the verification here because it serves to illustrate another general process by which differential resolvents can be calculated.

When $n=3$, we have (Art. 11)

$$\frac{dy}{dx} = -\frac{1}{3} \cdot \frac{y}{y-x},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{3^2} \cdot \frac{y^4}{(y-x)^3}.$$

And if we assume

$$\frac{d^2y}{dx^2} + \mu_1 \frac{dy}{dx} + \mu_2 y + \mu_3 = 0,$$

(where μ_1, μ_2, μ_3 are functions of x only which are to be determined shortly) substitute for $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ their values above given, clear of fractions, develope, reduce, &c. we are conducted to the equation

$$\begin{aligned} &3\,\{3^2\mu_3 x - 3^2\mu_2(1+x^2) - 2\mu_1 x + 1\}\,y^2 \\ &- \{3^3\mu_3(1+x^2) - 3^2\mu_2 x(11+x^2) - 3\mu_1(3+x^2) + 2x\}\,y \\ &+ 3\,\{3\mu_3 x(2+x^2) - 2.3^2\mu_2 x^2 - 2\mu_1 x\} = 0, \end{aligned}$$

which must vanish identically. Hence equating with zero the coefficients of the several powers of y, and solving with respect to μ, we find

$$\mu_1 = -\frac{x}{1-x^2},$$

$$\mu_2 = \frac{1}{3^2} \cdot \frac{1}{1-x^2},$$

$$\mu_3 = 0;^*$$

so that

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2} \cdot \frac{dy}{dx} + \frac{1}{3^2} \cdot \frac{1}{1-x^2} \cdot y = 0,$$

* This result might have been foreseen, and the process simplified accordingly. For since each of the roots y_1, y_2, y_3 must satisfy the differential resolvent, the sum of those roots must also satisfy it. But since the equation is wanting in its second term, the sum of the roots is zero, and consequently the resolvent must vanish identically on making $y=0$. The same reasoning applies, *mutatis mutandis*, to the higher equations, but not to the quadratic

$$y^2 - 2y + x = 0,$$

seeing that in this case the sum of the roots $= 2$, and the resolvent contains therefore a term independent of y.

which multiplied into $3^2(1-x^3)$ coincides with the equation calculated in the last article.

The same process is evidently applicable to the biquadratic

$$y^4 - 4y + 3x = 0,$$

for which ($n = 4$, see Art. 11)

$$\frac{dy}{dx} = -\frac{1}{4} \cdot \frac{y}{y-x},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4^2} \cdot \frac{y^5}{(y-x)^3},$$

$$\frac{d^3y}{dx^3} = -\frac{y^5}{4^3} \cdot \frac{10y-7x}{(y-x)^5},$$

and to the quintic

$$y^5 - 5y + 4x = 0,$$

for which ($n = 5$)

$$\frac{dy}{dx} = -\frac{1}{5} \cdot \frac{y}{y-x},$$

$$\frac{d^2y}{dx^2} = -\frac{1}{5^2} \cdot \frac{y^6}{(y-x)^3},$$

$$\frac{d^3y}{dx^3} = -\frac{3y^6}{5^3} \cdot \frac{4y-3x}{(y-x)^5},$$

$$\frac{d^4y}{dx^4} = -\frac{3y^6}{5^4} \cdot \frac{77y^6 - 114xy + 42x^2}{(y-x)^7}.$$

But the method first explained will be found upon the whole the more convenient of the two; and that therefore is the method which I shall follow in deducing the remaining resolvents.

15. Making $n = 4$, the formula at the foot of Art. 9 gives

$$\frac{dy}{dx} = -\frac{1}{4} \cdot \frac{1}{1-x^3} (y^3 + xy^2 + x^2y - 3),$$

or

$$4(1-x^3)\frac{dy}{dx} = -(y^3 + xy^2 + x^2y - 3).$$

Whence by differentiation and reduction, we have

$$4^2(1-x^3)^2\frac{d^2y}{dx^2} = -\{6x^2y^3 + (1+5x^3)y^2 + 3(1+x^3)xy - 18x^2\},$$

and $$4^3(1-x^3)^3\frac{d^3y}{dx^3} = -\{(43+65x^3)xy^3$$
$$+(61+47x^3)x^2y^2$$
$$+(10+77x^3+21x^6)y$$
$$-3(43+65x^3)x\}.$$

Combining the last three equations so as to eliminate y^3 and y^2, we find

$$4^3(1-x^3)\frac{d^3y}{dx^3} - 4^2.18x^2\frac{d^2y}{dx^2} - 4.43x\frac{dy}{dx} + 10y = 0,$$

or $$2^5(1-x^3)\frac{d^3y}{dx^3} - 2^4.3^2x^2\frac{d^2y}{dx^2} - 2.43x\frac{dy}{dx} + 5y = 0,$$

the differential resolvent for the biquadratic.

16. For the quintic ($n=5$, Art. 9) the formula gives

$$\frac{dy}{dx} = -\frac{1}{5}\cdot\frac{1}{1-x^4}(y^4+xy^3+x^2y^2+x^3y-4),$$

or $$5(1-x^4)\frac{dy}{dx} = -(y^4+xy^3+x^2y^2+x^3y-4),$$

and by successive differentiation, &c., we obtain

$$5^2(1-x^4)^2\frac{d^2y}{dx^2} = -\{10x^3y^4+(1+9x^4)y^3+(3+7x^4)xy^2$$
$$+2(3+2x^4)x^2y-40x^3\};$$

$$5^3(1-x^4)^3\frac{d^3y}{dx^3} = -\{15(9+11x^4)x^2y^4$$
$$+15(11+9x^4)x^3y^3$$
$$+3(4+67x^4+29x^8)y^2$$
$$+3(17+71x^4+12x^8)xy$$
$$-60(9+11x^4)x^2\},$$

and $$5^4(1-x^4)^4\frac{d^4y}{dx^4} = -\{75(17+134x^4+49x^8)xy^4$$
$$+60(31+173x^4+46x^8)x^2y^3$$
$$+30(121+328x^4+51x^8)x^3y^2$$
$$+3(77+2214x^4+2541x^8+168x^{12})y$$
$$-300(17+134x^4+49x^8)x\}.$$

Combining as before so as to eliminate all powers of y higher than the first, we find

$$5^4(1-x^4)\frac{d^4y}{dx^4}-2.5^5x^3\frac{d^3y}{dx^3}-3^2.5^3.13x^2\frac{d^2y}{dx^2}$$

$$-3.5^3.17x\frac{dy}{dx}+3.7.11y=0,$$

the differential resolvent for the quintic.

It is proper to mention that a large portion of the calculation of this resolvent was performed independently by Mr. Cockle, and that on comparing results and making one or two corrections, I found that his calculations coincided (so far as they went, for they only extended to the determination of the differential coefficients) with my own.

17. If the differential resolvents be employed (as we shall hereafter see they may be) as tests of existing theories of differential equations, it is obviously of the utmost importance that their accuracy be placed beyond dispute. I have therefore verified all the foregoing results with the greatest care. The process of verification I here proceed to illustrate by an example. Such details are not without their value.

If we assume $y=-x$, the equation

$$y^n-ny+(n-1)x=0$$

becomes

$$(y^{n-1}-2n+1)\,y=0;$$

and neglecting the root zero, we have

$$y=-x=(2n-1)^{\frac{1}{n-1}}.$$

Now substituting this value in the formulæ for the several differential coefficients exhibited in Art. 11, there results

$$\frac{dy}{dx}=-\frac{1}{2n},$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2^3n^2}(2n-1)^{\frac{n-3}{n-1}},$$

$$\frac{d^3y}{dx^3}=-\frac{1}{2^5n^3}(4n+1)(2n-1)^{\frac{n-5}{n-1}},$$

$$\frac{d^4y}{dx^4}=-\frac{1}{2^7n^4}(24n^2+20n-1)(2n-1)^{\frac{n-4}{n-1}}.$$

In order therefore to verify the expressions for the several differential coefficients found in the last five articles, we have only to substitute for y, or $-x$, the numerical values $3, 5^{\frac{1}{2}}, 7^{\frac{1}{3}}, 3^{\frac{1}{2}}$ successively, and compare the results with those obtained by making n equal to 2, 3, 4, 5 successively in the above formulæ. Thus, for the quintic (*ex. gr.*), we have, by either set of substitutions,

$$\frac{dy}{dx} = -\frac{1}{2.5}, \quad \frac{d^2y}{dx^2} = -\frac{3^{\frac{1}{2}}}{2^3.5^2}, \quad \frac{d^3y}{dx^3} = -\frac{3^2.7}{2^5.5^3}, \quad \frac{d^4y}{dx^4} = -\frac{3^{\frac{1}{2}}.233}{2^7.5^4};$$

and the sinister member of the resolvent, making $y = -x = 3^{\frac{1}{2}}$, becomes

$$\frac{3^{\frac{1}{2}}}{2^4}(699 - 4725 + 10530 - 10200 + 3696),$$

which is equal to zero, as it ought to be. Other verifications are indicated in Art. 8, but it does not seem necessary to dwell on them here.

18. The differential resolvents which we have calculated may be all, with the exception of that for the quadratic, comprehended under one general symbolical form. For, transforming by Dr. Boole's process (see his Memoir on a "General Method in Analysis," in the *Philosophical Transactions*, for 1844, Part II., or his *Treatise on Differential Equations*, Chapter XVII.) in which ε^θ is substituted for x and D represents the differential symbol $\frac{d}{d\theta}$, we are conducted to the following results, viz. For the quadratic, the Boolian or symbolical form of the resolvent is

$$y - \frac{D - \frac{3}{2}}{D}\,\varepsilon^\theta y = \frac{1}{2D}\,\varepsilon^\theta.$$

For the cubic, it is

$$y - \frac{(D - \frac{5}{3})(D - \frac{7}{3})}{D(D-1)}\,\varepsilon^{2\theta} y = 0.$$

For the biquadratic, it is

$$y - \frac{(D - \frac{7}{4})(D - \frac{10}{4})(D - \frac{13}{4})}{D(D-1)(D-2)}\,\varepsilon^{3\theta} y = 0.$$

For the quintic, it is

$$y - \frac{(D - \frac{9}{5})(D - \frac{13}{5})(D - \frac{17}{5})(D - \frac{21}{5})}{D(D-1)(D-2)(D-3)}\,\varepsilon^{4\theta} y = 0.$$

So that the general form, suggested by induction from the above, is

$$y-\frac{\left(D-\frac{2n-1}{n}\right)\left(D-\frac{3n-2}{n}\right)\left(D-\frac{4n-3}{n}\right)\cdots\left(D-\frac{n^2-n+1}{n}\right)}{D(D-1)(D-2)\cdots(D-n+2)}\varepsilon^{(n-1)\theta}y=0,$$

the only exception being the resolvent for the quadratic which, for reasons given in the footnote under article 14, must contain a term independent of y, and therefore cannot be reduced to a binomial form. It is worthy of notice however that the sinister member of that resolvent coincides with what the sinistic member of the general equation becomes when we make $n=2$. It is also worthy of notice that the fractions

$$\frac{2n-1}{n},\ \frac{3n-2}{n},\ \frac{4n-3}{n},\ \cdots\frac{n^2-n+1}{n},$$

which occur in the general equation, are in arithmetical progression, the common difference being $\dfrac{n-1}{n}$.

19. The general formula exhibited in the last article was obtained by induction from the cases $n=3$, $n=4$, $n=5$. In order to complete the induction and shew that the formula holds universally, it would be necessary to prove that, if it is true for $n=m$, it is also true for $n=m+1$. The above induction however is sufficiently wide for our present purpose, inasmuch as when n is greater than five, the given equation cannot in general be reduced to the trinomial form with which we are now working. We might therefore be content with the theorem as here established. But Mr. Cayley, in a paper entitled "Note on a Differential Equation," read before the Literary and Philosophical Society of Manchester, February 18, 1862, has, in a remarkably beautiful analysis, arrived at a result by means of which it is easy to establish the theorem in all its generality. A brief abstract of Mr. Cayley's paper is here, for convenience, transcribed from p. 193 of the current volume of the *Manchester Society's Proceedings;* it is as follows:

"The investigation was suggested by Mr. Harley's remarks* on the Theory of the Transcendental Solution of

* The remarks to which Mr. Cayley here refers occur at pp. 181-184 of the same volume. Some supplementary observations of mine on the same subject will also be found at pp. 199-201. In these two com-

Algebraic Equations, communicated to the Society at the Meeting of the 4th February last.

The equation $y = u + ay^n$ (which is used instead of Mr. Harley's equation $y^n - ny + (n-1)x = 0$) gives, by Lagrange's theorem an expression for y in the form of an infinite series, and by means of this series it is shewn that y satisfies the differential equation

$$\left[u\frac{d}{du}\right]^{n-1} y = na\left[\frac{n}{n-1}u\frac{d}{du} - \frac{2n-1}{n-1}\right]^{n-1} u^{n-1}y.$$

($[m]^\nu$ denotes as usual the factorial

$$m(m-1)(m-2)\ldots(m-\nu+1)).$$

It is remarked that the equation may be written in the form

$$\left\{\alpha_0 + \alpha_1 u\frac{d}{du} + \alpha_2 u^2\left(\frac{d}{du}\right)^2 \ldots + \alpha_{n-1}u^{n-1}\left(\frac{d}{du}\right)^{n-1}\right\} y$$

$$= \frac{1}{na}\left(\frac{d}{du}\right)^{n-1} y,$$

and the law of the coefficients is obtained."

Mr. Cayley's equation

$$y = u + ay^n$$

becomes identical with that which I have employed

$$y^n - ny + (n-1)x = 0$$

on putting

$$u = \frac{n-1}{n}x, \quad a = \frac{1}{n};$$

and these substitutions being made for u and a in Mr. Cayley's differential equation

$$\left[u\frac{d}{du}\right]^{n-1} y = na\left[\frac{n}{n-1}u\frac{d}{du} - \frac{2n-1}{n-1}\right]^{n-1} u^{n-1}y,$$

it becomes

$$\left[x\frac{d}{dx}\right]^{n-1} y = \left[\frac{n}{n-1}x\frac{d}{dx} - \frac{2n-1}{n-1}\right]^{n-1}\left(\frac{n-1}{n}x\right)^{n-1} y,$$

munications I gave (without proof) several of the more remarkable results investigated in the present paper. I may add that at pp. 202–203 of the same volume there appear some remarks on Transcendental Solution by Mr. Cockle (dated Temple, February, 1862), which I had the honor of laying before the Manchester Society along with the last-named communication from myself.—R. H.

or, what is the same thing,

$$\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)\ldots\left(x\frac{d}{dx}-n+2\right)y$$

$$=\left(x\frac{d}{dx}-\frac{2n-1}{n}\right)\left(x\frac{d}{dx}-\frac{3n-2}{n}\right)\left(x\frac{d}{dx}-\frac{4n-3}{n}\right)$$

$$\ldots\left(x\frac{d}{dx}-\frac{n^2-n+1}{n}\right)x^{n-1}y\,;$$

whence, writing ε^θ in place of x, we have

$$\frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\left(\frac{d}{d\theta}-2\right)\ldots\left(\frac{d}{d\theta}-n+2\right)y$$

$$=\left(\frac{d}{d\theta}-\frac{2n-1}{n}\right)\left(\frac{d}{d\theta}-\frac{3n-2}{n}\right)\left(\frac{d}{d\theta}-\frac{4n-3}{n}\right)$$

$$\ldots\left(\frac{d}{d\theta}-\frac{n^2-n+1}{n}\right)\varepsilon^{(n-1)\theta}y,$$

which verifies the form given in the last article and establishes the theorem in all its universality.

I may notice here that the general resolvent equation for the trinomial may be written, as is easily seen, under the following form, viz.

$$[D]^{n-1}y=\left(\frac{n-1}{n}\right)^{n-1}[\delta]^{n-1}\varepsilon^{(n-1)\theta}y,$$

where δ and D are connected by the defining relation

$$(n-1)\,\delta=nD-2n+1.$$

20. I next proceed to apply the theory to the solution of the equation in y. And here again I begin with the simplest case $(n=2)$

$$y^2-2y+x=0,$$

for which (Art. 12)

$$2(1-x)\frac{dy}{dx}+y-1=0,$$

and therefore, by a known process,

$$y=1+c\varepsilon^{-\int\frac{dx}{2(1-x)}}$$

$$=1+c\varepsilon^{\frac{1}{2}\log_\varepsilon(x-1)}$$

$$=1+c\,(x-1)^{\frac{1}{2}}.$$

In order to determine the constant, make, in the given quadratic, $y=2$, then $x=0$, and therefore

$$2=1+c(-1)^{\frac{1}{2}}, \text{ or } c=(-1)^{\frac{1}{2}}.$$

Consequently $\quad y=1+(1-x)^{\frac{1}{2}},$

the usual algebraic solution.

21. The cubic resolvent

$$3^2(1-x^2)\frac{d^2y}{dx^2}-3^2x\frac{dy}{dx}+y=0$$

is of a known integrable form. In fact, following the usual method and changing the independent variable by assuming $x=\sin t$, we find

$$\frac{d^2y}{dt^2}+\frac{1}{3^2}y=0,$$

and the final solution is

$$y=c_1\sin\frac{t+c_2}{3}$$

$$=c_1\sin\frac{\sin^{-1}x+c_2}{3}.$$

It is to be noticed that if we were to regard and treat this value not as that of y but of a constituent of y, we should be remitted to the same final result; for the sum of any number of functions of the form

$$c_1\sin\frac{\sin^{-1}x+c_2}{3}$$

is obviously a function of the same form. The above may therefore be dealt with as the complete solution.

Perhaps the most convenient method of determining the arbitrary constants is the following: When $x=0$, $y=0$, $\sqrt{(3)}$ or $-\sqrt{(3)}$; and since $\sin^{-1}0=0$, π or $-\pi$, we may write

$$0=c_1\sin\frac{c_2}{3},$$

$$\sqrt{(3)}=c_1\sin\frac{\pi+c_2}{3},$$

$$-\sqrt{(3)}=c_1\sin\frac{-\pi+c_2}{3};$$

whence, combining any two of these conditions, we find

$$c_1=2, \text{ and } c_2=2r\pi,$$

r being zero or any integer. We have therefore finally

$$y = 2 \sin \frac{\sin^{-1} x + 2r\pi}{3}.$$

Mr. Cockle has shewn* that the differential equation now under discussion may be integrated without any change of the independent variable. For the equation may be put under the symbolical form

$$\left\{\sqrt{(1-x^2)} \cdot \frac{d}{dx} + m\right\} \left\{\sqrt{(1-x^2)} \cdot \frac{d}{dx} - m\right\} y = 0,$$

where $m = \frac{1}{3}\sqrt{(-1)}$; and we have therefore

$$y = \left\{\sqrt{(1-x^2)} \cdot \frac{d}{dx} - m\right\}^{-1} \left\{\sqrt{(1-x^2)} \cdot \frac{d}{dx} + m\right\}^{-1} 0$$

$$= \left\{\sqrt{(1-x^2)} \cdot \frac{d}{dx} - m\right\}^{-1} k\varepsilon^{-m \sin^{-1} x}$$

$$= k_1 \varepsilon^{m \sin^{-1} x} + k_2 \varepsilon^{-m \sin^{-1} x},$$

or, restoring the numerical value of m,

$$y = k_1 \varepsilon^{\frac{1}{3}\sqrt{(-1)} \cdot \sin^{-1} x} + k_2 \varepsilon^{-\frac{1}{3}\sqrt{(-1)} \cdot \sin^{-1} x}$$

$$= c_1 \sin \frac{\sin^{-1} x + c_2}{3}$$

as before.

In each of the foregoing solutions x is tacitly assumed not greater than unity, so that the discussion embraces only "the irreducible case." But we may avoid the restriction and generalize the solution by employing (in the integrations) logarithmic in place of trigonometric forms. Thus the resolvent may be written

$$\left\{\sqrt{(x^2-1)} \cdot \frac{d}{dx} + m\right\} \left\{\sqrt{(x^2-1)} \cdot \frac{d}{dx} - m\right\} y = 0,$$

where $m = \frac{1}{3}$; and consequently

$$y = \left\{\sqrt{(x^2-1)} \cdot \frac{d}{dx} - m\right\}^{-1} \left\{\sqrt{(x^2-1)} \cdot \frac{d}{dx} + m\right\}^{-1} 0$$

$$= \left\{\sqrt{(x^2-1)} \cdot \frac{d}{dx} - m\right\}^{-1} c\,\{x + \sqrt{(x^2-1)}\}^{-m}$$

$$= c_1 \{x + \sqrt{(x^2-1)}\}^{m} + c_2 \{x + \sqrt{(x^2-1)}\}^{-m}$$

* See Mr. Cockle's paper "On Transcendental and Algebraic Solution. Supplementary Paper." *Philosophical Magazine*, for February, 1862.

(which, restoring the value of m)

$$= c_1\{x+\sqrt{(x^2-1)}\}^{\frac{1}{3}} + \frac{c_2}{\{x+\sqrt{(x^2-1)}\}^{\frac{1}{3}}},$$

the complete solution. For the determination of the constants, let (as before) $x=0$, then $y=0$, $\sqrt{(3)}$ or $-\sqrt{(3)}$; and if ω denote as usual one of the unreal cube roots of unity, say

$$-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(-3)},$$

and i the square root of negative unity, then since

$$\frac{1}{i} = -i = \omega^r \sqrt[3]{(i)},$$

r being 0, 1 or 2, we may write

$$\begin{aligned} 0 &= -c_1 i + c_2 i, \\ \sqrt{(3)} &= -c_1 i\omega + c_2 i\omega^2, \\ -\sqrt{(3)} &= -c_1 i\omega^2 + c_2 i\omega; \end{aligned}$$

from any two of which conditions we obtain

$$c_1 = c_2 = -1;$$

so that the roots of the cubic are all included in the formula

$$\omega^r\{-x-\sqrt{(x^2-1)}\}^{\frac{1}{3}} + \frac{1}{\omega^r\{-x-\sqrt{(x^2-1)}\}^{\frac{1}{3}}},$$

or (what is the same thing) in

$$\omega^r\{-x+\sqrt{(x^2-1)}\}^{\frac{1}{3}} + \frac{1}{\omega^r\{-x+\sqrt{(x^2-1)}\}^{\frac{1}{3}}}.$$

This agrees with known results.

22. The process of decomposition employed in the preceding article in dealing with the cubic resolvent may be applied with effect to the more general equation

$$(1+ax^2)\frac{d^2y}{dx^2} + ax\frac{dy}{dx} \pm n^2 y = 0.$$

For, writing m^2 in place of $\mp n^2$, we may deduce successively

$$\left\{\sqrt{(1+ax^2)}\,.\frac{d}{dx}+m\right\}\left\{\sqrt{(1+ax^2)}\,.\frac{d}{dx}-m\right\}y = 0;$$

$$\left\{\sqrt{(1+ax^2)}\,.\,\frac{d}{dx}-m\right\}y=\left\{\sqrt{(1+ax^2)}\,.\,\frac{d}{dx}+m\right\}^{-1}0$$

$$=c\,\{x\sqrt{(a)}+\sqrt{(1+ax^2)}\}^{-\frac{m}{\sqrt{(a)}}};$$

$$y=c\left\{\sqrt{(1+ax^2)}\,.\,\frac{d}{dx}+m\right\}^{-1}\{x\sqrt{(a)}+\sqrt{(1+ax^2)}\}^{-\frac{m}{\sqrt{(a)}}}$$

$$=c_1\,\{x\sqrt{(a)}+\sqrt{(1+ax^2)}\}^{\frac{m}{\sqrt{(a)}}}+c_2\,\{x\sqrt{(a)}+\sqrt{(1+ax^2)}\}^{-\frac{m}{\sqrt{(a)}}}.$$

The allied form

$$x^2\,(x^2+a)\,\frac{d^2y}{dx^2}+x\,(2x^2+a)\,\frac{dy}{dx}\pm n^2y=0$$

noticed by Dr. Boole in his work on *Differential Equations*, at p. 419, may be decomposed and solved in a similar manner. It is in fact equivalent to

$$\left\{x\sqrt{(x^2+a)}\,.\,\frac{d}{dx}+m\right\}\left\{x\sqrt{(x^2+a)}\,.\,\frac{d}{dx}-m\right\}y=0,$$

which by successive integrations gives

$$y=c_1\{-x\sqrt{(a)}+\sqrt{(1+ax^2)}\}^{\frac{m}{\sqrt{(a)}}}+c_2\{-x\sqrt{(a)}+\sqrt{(1+ax^2)}\}^{-\frac{m}{\sqrt{(a)}}},$$

a result which might have been more readily obtained by simply writing $-x$ in place of x in the above solution.

23. Before leaving the cubic equation it may be worth while to indicate its solution by the process given in Mr. Cockle's "Note on Transcendental Roots." If we eliminate y between the given equation

$$y^3-3y+2x=0,$$

and its first derivative

$$\frac{dy}{dx}=-\tfrac{1}{3}\,.\,\frac{y}{y-x},$$

we are led to

$$3^2\,(1-x^2)\left(\frac{dy}{dx}\right)^2-3^2\left(\frac{dy}{dx}\right)-2=0,$$

which may be immediately transformed into

$$y'^3-3y'+2x'=0,$$

where $\qquad y'=-3\sqrt{(1-x^2)}\,.\,\dfrac{dy}{dx},$

^{2}).

This equation is of the same form as the original one; so that if we assume $y = \phi x$, we shall have $y' = \phi x'$; or, what is the same thing,

$$-3\sqrt{(1-x^2)}\,.\,\frac{d\phi x}{dx} = \phi\sqrt{(1-x^2)},$$

that is to say

$$\frac{d\phi x}{dx} = -\frac{\phi\sqrt{(1-x^2)}}{3\sqrt{(1-x^2)}},*$$

and therefore
$$\frac{d\phi\sqrt{(1-x^2)}}{d\sqrt{(1-x^2)}} = -\frac{\phi x}{3x}.$$

By differentiation we obtain

$$\frac{d^2\phi x}{dx^2} = -\frac{x\phi\sqrt{(1-x^2)}}{3(1-x^2)^{\frac{3}{2}}} - \frac{1}{3\sqrt{(1-x^2)}}\,.\,\frac{d\phi\sqrt{(1-x^2)}}{dx}$$

$$= -\frac{x\phi\sqrt{(1-x^2)}}{3(1-x^2)^{\frac{3}{2}}} - \frac{1}{3\sqrt{(1-x^2)}}\,.\,\frac{d\phi\sqrt{(1-x^2)}}{d\sqrt{(1-x^2)}}\,.\,\frac{d\sqrt{(1-x^2)}}{dx}$$

$$= -\frac{x\phi\sqrt{(1-x^2)}}{3(1-x^2)^{\frac{3}{2}}} - \frac{1}{3\sqrt{(1-x^2)}}\,.\left(-\frac{\phi x}{3x}\right).\,\frac{-x}{\sqrt{(1-x^2)}}$$

$$= \frac{x}{1-x^2}\,.\,\frac{d\phi x}{dx} - \frac{\phi x}{3^2(1-x^2)};$$

whence
$$\frac{d^2\phi x}{dx^2} - \frac{x}{1-x^2}\,.\,\frac{d\phi x}{dx} + \frac{\phi x}{3^2(1-x^2)} = 0,$$

* I notice that since

$$\frac{d\phi x}{dx} = -\tfrac{1}{3}\,.\,\frac{y}{y-x} = -\tfrac{1}{3}\,.\,\frac{\phi x}{\phi x - x},$$

therefore we have

$$\frac{\phi x}{\phi x - x} = \frac{\phi\sqrt{(1-x^2)}}{\sqrt{(1-x^2)}},$$

and consequently

$$\frac{x}{\phi x} + \frac{\sqrt{(1-x^2)}}{\phi\sqrt{(1-x^2)}} = 1,$$

a functional equation which admits of a variety of solutions, such as

$$\phi x = 2x,\quad \phi x = \frac{1}{x},\ \&c.,$$

which are wholly incongruous with the fundamental condition

$$(\phi x)^3 - 3\phi x + 2x = 0.$$

a result identical with that obtained by the process which Mr. Cockle gave in his first paper "On Transcendental and Algebraic Solution." The latter process, which is perfectly general in its form and character, has been followed in the present Memoir.

Castle Hill House, Brighouse, Yorkshire,
May 26, 1862.

(To be continued).

THEOREMS CONCERNING WAVE-VELOCITIES AND RAY-SLOWNESSES IN A BIAXAL CRYSTAL.

By William Walton, M.A., Trinity College.

THE following theorems, the demonstration of which is extremely easy, may perhaps be of some interest to Students.

In any three directions, at right angles to each other, each of the following expressions is invariable; viz.

(1) The sum of the squares of the six wave-velocities:

(2) The sum of the squares of the three rectangles between the conjugate wave-velocities:

(3) The sum of the squares of the six ray-slownesses:

(4) The sum of the squares of the three rectangles between the concurrent ray-slownesses.

If l, m, n, be the direction-cosines of a wave-velocity v in a biaxal crystal,

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0.$$

Clearing the equation of fractions, we have

$$v^4-\{l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)\}\,v^2$$
$$+\,l^2b^2c^2+m^2c^2a^2+n^2a^2b^2=0.$$

Let v_1^2, v_2^2, be the two values of v^2 in this quadratic: then

$$v_1^2+v_2^2=l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2).$$

Similarly, for two other directions of wave-velocity,

$$v_1'^2 + v_2'^2 = l'^2 (b^2 + c^2) + m'^2 (c^2 + a^2) + n'^2 (a^2 + b^2),$$

$$v_1''^2 + v_2''^2 = l''^2 (b^2 + c^2) + m''^2 (c^2 + a^2) + n''^2 (a^2 + b^2).$$

Suppose these three directions to be at right angles to each other: then, adding together the last three equations, we get

$$\Sigma (v_1^2 + v_2^2) = 2 (a^2 + b^2 + c^2) \text{ .... } (1).$$

Again, reverting to the quadratic in v^2, we see that

$$v_1^2 v_2^2 = l^2 b^2 c^2 + m^2 c^2 a^2 + n^2 a^2 b^2,$$

and therefore, summing in relation to three directions at right angles to each other, we have

$$\Sigma (v_1 v_2)^2 = b^2 c^2 + c^2 a^2 + a^2 b^2 \text{ .... } (2).$$

Finally, in virtue of the reciprocal relations between wave-velocities and ray-slownesses, we may replace a, b, c, v_1, v_2, respectively, by $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, ρ_1, ρ_2, where ρ_1, ρ_2, represent two ray-slownesses in any one direction: hence, referring to (1) and (2), we see that

$$\Sigma (\rho_1^2 + \rho_2^2) = 2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \text{ .... } (3),$$

and

$$\Sigma (\rho_1 \rho_2)^2 = \frac{1}{b^2 c^2} + \frac{1}{c^2 a^2} + \frac{1}{a^2 b^2} \text{ .... } (4).$$

Cor. From the equations (1), (2), (3), (4), we see that

$$\frac{\Sigma (v_1^2 + v_2^2)}{\Sigma (v_1 v_2)^2} \cdot \frac{\Sigma (\rho_1^2 + \rho_2^2)}{\Sigma (\rho_1 \rho_2)^2} = 4;$$

a relation not involving a, b, c, and therefore invariable for all biaxal crystals.

September, 1861.

GEOMETRICAL THEOREMS.

By the Rev. George Salmon.

In this *Journal*, Vol. IV., p. 152, I gave expressions for the distances from the intersection of perpendiculars of a triangle to the centres of the inscribed and circumscribing circle, and thence deduced the locus of intersection of perpendiculars when the two circles are given. Dr. Hart subsequently extended one of my theorems to the case of a conic, and Mr. Burnside has since extended the other, and has deduced the locus of the intersection of perpendiculars of a triangle inscribed in one conic and circumscribed about another. I noticed lately that a proof which I published in Terquem's *Annales*, of a theorem of Captain Faure's, virtually contains the proof of Dr. Hart's and Mr. Burnside's theorems. I therefore reproduce that proof here.

The foundation of the proof is the principle (see my *Conics*) that if U and V are two conics, and if the discriminant of $U+\lambda V$ be $\Delta+\lambda\Theta+\lambda^2\Theta'+\lambda^3\Delta'$; then the coefficients Δ, Θ, Θ', Δ' are invariants. In fact since if $U+\lambda V$ represent right lines, it will continue to represent right lines, no matter how the co-ordinates be transformed, the roots of the cubic in λ just written, and therefore the mutual ratios of the coefficients of the equation will be unchanged by transformation of co-ordinates.

If $$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2,$$
then $$\Theta = a'(bc-f^2) + b'(ca-g^2) + c'(ab-h^2) + 2f'(gh-af) + 2g'(hf-bg) + 2h'(fg-ch).$$

We get Δ' and Θ' by interchanging the accented and unaccented letters.

If now a triangle self-conjugate with regard to U be inscribed in V, Θ will vanish. For if this triangle be taken for the triangle of reference, U will take the form $ax^2+by^2+cz^2$ and V the form $f'yz+g'zx+h'xy$; or, in other words, we have a', b', c', f, g, h all vanishing; in which case it is easily seen that Θ vanishes; and since it vanishes for these particular trilinear co-ordinates, it vanishes because it is an invariant for every co-ordinate.

In like manner Θ will vanish if a triangle self-conjugate with regard to V be circumscribed about U; for the conditions that the triangle shall be self-conjugate are f', g', h' all $=0$; and that it shall circumscribe U are $bc=f'^2$, $ca=g'^2$, $ab=h'^2$, suppositions which make Θ vanish.

So again $\Theta'=0$ is the condition either that a triangle self-conjugate with regard to V shall be inscribed in U, or that a triangle self-conjugate with regard to U shall circumscribe V. These theorems are stated, *Conics*, but are repeated here for the convenience of the reader.

Let U now be the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and V the circle

$$(x-\alpha)^2+(y-\beta)^2=r^2,$$

and let us form the discriminant of $U+\lambda V$, and we find without difficulty

$$\Theta=\frac{1}{a^2b^2}\left(\alpha^2+\beta^2-r^2-a^2-b^2\right),$$

$$\Theta'=\frac{\alpha^2-r^2}{a^2}+\frac{\beta^2-r^2}{b^2}-1,$$

$\Theta=0$ gives a relation between the radius of the circle circumscribing any self-conjugate triangle with regard to an ellipse, and the distance of its centre from the centre of the conic. Captain Faure had stated this relation in the form "The tangent drawn from the centre of the ellipse to this circle is equal to the chord of the quadrant of the ellipse." Mr. Gaskin had stated it, "The circle circumscribing a self-conjugate triangle cuts at right angles the circle which is the locus of intersection of rectangular tangents to the conic." It is easily seen that $\Theta=0$ is equivalent to either of these statements. I gave in *Terquem* some inferences from the equation $\Theta'=0$ which gives a relation between the co-ordinates of the centre and the radius of a circle *inscribed* in a self-conjugate triangle.

But what I wish now to point out is that $\Theta'=0$ is also the condition that a triangle self-conjugate with regard to the circle may be inscribed in the ellipse. Now if a triangle be self-conjugate with respect to a circle, the centre of the circle is the intersection of perpendiculars of the triangle; and the rectangle under the segments of any perpendicular

is equal to $-r^2$ when the triangle is acute angled and r^2 if it be obtuse angled. The equation then $\Theta' = 0$ or

$$\left(\frac{1}{a^2} + \frac{1}{b^2}\right) r^2 = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1$$

gives us Dr. Hart's theorem. "If a triangle be inscribed in a conic the rectangle under the segments of perpendiculars depends only on the position of the intersection of the perpendiculars, and is the same while that intersection moves on a conic similar and concentric with the given conic."

In like manner from the equation $\Theta = 0$ or $r^2 = \alpha^2 + \beta^2 - a^2 - b^2$ we get Mr. Burnside's theorem that "If a triangle be circumscribed about a conic the rectangle under the segments of a perpendicular depends only on the distance of the centre from the intersection of perpendiculars." Since this expression shows that the triangle is acute angled or obtuse angled according as the intersection of perpendiculars is within or without the circle $x^2 + y^2 = a^2 + b^2$, we could infer thus that this circle is the locus of intersection of rectangular tangents.

If a triangle be inscribed in the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and circumscribed about one, the co-ordinates of whose centre are α', β' and axes A, B; then by equating the two values of r^2 found from the consideration of the two triangles, we infer that the co-ordinates of the intersection of perpendiculars satisfy the equation

$$\frac{a^2b^2}{a^2+b^2}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) = (\alpha - \alpha')^2 + (\beta - \beta')^2 - A^2 - B^2.$$

The locus is therefore a conic passing through the points where one conic is met by the locus of rectangular tangents to the other.

This theorem admits of a further generalization, "If a triangle be inscribed in a conic U and circumscribed about another V its *pole* with regard to a third conic W moves on a conic and its *polar* envelopes a conic." By the pole of a triangle with respect to a conic, I mean the point through which pass the lines joining the vertices of the triangle to those of its conjugate, and by the polar the line joining the intersections of corresponding sides of the two triangles. To prove this, it is to be observed that if a triangle be self-conjugate with regard to any conic, its polar and pole with regard to a conic having double contact with this are the chord of contact and it[illegible] proof of this is obvious.

If now we form the condition that a triangle inscribed in U should be self-conjugate with regard to $W+(ax+\beta y+\gamma z)^2$, we get a condition of the form $\Theta+(A\alpha^2+\&c.)=0$, where $A\alpha^2+\&c.$ is the condition that $ax+\beta y+\gamma z$ should touch U. Again if we form the condition that a triangle circumscribed about V should be self-conjugate with regard to $W+(ax+\beta y+\gamma z)^2$, we get a condition of the form $\Theta'+(A'\alpha^2+\&c.)$ and therefore when both conditions are fulfilled, we have a relation of the form

$$\Theta'(A\alpha^2+\&c.)=\Theta(A'\alpha^2+\&c.)$$

which proves that $ax+\beta y+\gamma z$ envelopes a conic, and that its pole moves on another.

Trinity College, Dublin,
June 26, 1862.

NOTE ON CERTAIN CURVES OF THE THIRD DEGREE.

By Michael Roberts.

In an interesting paper which has appeared in this *Journal* (Vol. v., pp. 54–58) Mr. Samuel Roberts has investigated the properties of certain systems of curves of the third degree. In the few remarks which follow I propose to consider the same curves as affording an illustration of the general principles of the theory of cubics as laid down by Mr. Salmon in the ninth section of the third chapter of his remarkable work on the *Higher Plane Curves.* The curves in question have for equation in trilinear coordinates

$$k_1\left\{\left(\frac{\beta}{n}\right)^2-\left(\frac{\gamma}{p}\right)^2\right\}\frac{\alpha}{m}+k_2\left\{\left(\frac{\gamma}{p}\right)^2-\left(\frac{\alpha}{m}\right)^2\right\}\frac{\beta}{n}$$
$$+k_3\left\{\left(\frac{\alpha}{m}\right)^2-\left(\frac{\beta}{n}\right)^2\right\}\frac{\gamma}{p}=0.$$

Denoting then by S and T the quartinvariant and the sextinvariant of the above form I find

$$S=\frac{1}{m^4n^4p^4}\{k_1^4+k_2^4+k_3^4-k_1^2k_2^2-k_1^2k_3^2-k_2^2k_3^2,$$

or, if ω is an imaginary cube root of unity,

$$S = \frac{1}{m^4 n^4 p^4} \{(k_1^2 + \omega k_2^2 + \omega^2 k_3^2)(k_1^2 + \omega^2 k_2^2 + \omega k_3^2)\},$$

$$T = -\frac{4}{m^6 n^6 p^6} \{(2k_1^2 - k_2^2 - k_3^2)(2k_2^2 - k_1^2 - k_3^2)(2k_3^2 - k_1^2 - k_2^2)\}.$$

Let us put

$$L = k_1^2 + \omega k_2^2 + \omega^2 k_3^2, \quad M = k_1^2 + \omega^2 k_2^2 + \omega k_3^2,$$

we then find

$$T = -\frac{4}{m^6 n^6 p^6} \{L^3 + M^3\},$$

and the discriminant

$$R = 64S^3 - T^2 = -\frac{16}{m^{12} n^{12} p^{12}} \{L^3 - M^3\}^2,$$

which in this case assumes the remarkable form of a perfect square. We deduce from the last equation

$$\sqrt{(-R)} = \frac{4}{m^6 n^6 p^6} \{(L - M)(L - \omega M)(L - \omega^2 M)\}.$$

But $L - M = \omega(1-\omega)(k_2^2 - k_3^2)$, $L - \omega M = (1-\omega)(k_1^2 - k_2^2)$,

$$L - \omega^2 M = (1 - \omega^2)(k_1^2 - k_3^2).$$

Hence $m^6 n^6 p^6 \sqrt{(-R)} = 3\omega(1-\omega)(k_1^2 - k_2^2)(k_1^2 - k_3^2)(k_2^2 - k_3^2)$.

The curve has therefore a double point if two of the three quantities k_1^2, k_2^2, k_3^2 become equal, and it is cusped if these three quantities become equal. If they are in arithmetic progression, the cubic becomes of the class called by Mr. Salmon *harmonic* (*Higher Plane Curves*, p. 193), as in this case the tangents drawn to the curve from any point in it form an harmonic pencil; and the cubic coincides with its second Hessian. To find the curve whose Hessian shall be the given cubic, we must solve the equation $\mu^3 - \frac{1}{3}S\mu - \frac{T}{108} = 0$ (*Higher Plane Curves*, p. 192) whose roots μ_1, μ_2, μ_3 are

$$\mu_1 = \frac{L+M}{3m^2 n^2 p^2}, \quad \mu_2 = \frac{\omega L + \omega^2 M}{3m^2 n^2 p^2}, \quad \mu_3 = \frac{\omega^2 L + \omega M}{3m^2 n^2 p^2}.$$

15, Trinity College, Dublin,
August 28, 1862.

THEOREMS ON LINES OF CURVATURE AND GEODESIC LINES ON AN ELLIPSOID.

By J. G. LAING, B.A., St. John's College.

TWO *geodesic lines TP, TQ are drawn from any point T on the surface of an ellipsoid to touch any line of curvature in P and Q. Prove that the internal and external bisectors of the angle PTQ are tangents to the lines of curvature passing through the point T.*

With the usual notation, we have along the line of curvature

$$pD = c \text{ (a constant)},$$

also along TP

$$pD = c';$$

but at P, p and D are the same for TP and the line of curvature, and therefore $c' = c$.

Similarly along TQ, we have $pD = c$.

Now at T, p is the same for TP and TQ, and therefore D is the same for both, or the tangents at T to TP, TQ are parallel to equal diameters of the diametral section of the ellipsoid. Hence the tangents to the lines of curvature at T, which are parallel to the axes of the diametral section, are the internal and external bisectors of the angle PTQ.

Q. E. D.

COR. If the umbilici U, V be joined to T by geodesic lines, we know that the tangents to the lines of curvature at T bisect the angles between UT and VT, and therefore, by what has been proved above, it follows that angle UTP = angle VTQ.

It is proved in Salmon's *Geometry of Three Dimensions*, Art. 198, that if a sphero-conic be projected on either plane of circular section, by lines parallel to the least axis of the ellipsoid, the projection will be a circle.

If the equations to the sphero-conic be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$x^2 + y^2 + z^2 = r^2,$$

the equation to the projection is

$$x^2 + y^2 = \frac{b^2}{b^2 - c^2}(r^2 - c^2) \text{ .... (I),}$$

the axes being the axis of y, and an axis, in the plane of circular section, at right angles to it.

If we project the lines of curvature in a similar manner, the projections will be confocal conics, the projections of the umbilici being the foci. If a line of curvature be determined by the equations

$$\left.\begin{array}{c} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ \dfrac{x^2}{a^2 - \lambda^2} + \dfrac{y^2}{b^2 - \lambda^2} + \dfrac{z^2}{c^2 - \lambda^2} = 1 \end{array}\right\}, *$$

we get, for the projection on the circular section, the conic

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = \frac{b^2}{b^2 - c^2} \text{ .... (II).}$$

I have not seen the following propositions stated elsewhere; they may perhaps be new:

Prop. 1. *If from an umbilicus we draw a geodesic line perpendicular to any geodesic tangent to a line of curvature, the locus of the foot of the perpendicular is a sphero-conic, which projects into the circle which is the locus of the foot of the perpendicular from the focus on the tangent to the projection of the line of curvature.*

Let UY be the geodesic through the umbilicus U at right angles to the geodesic tangent PY, then along UY we have $pD = ac$, and along PY, $pD' = \mu$ (the constant of the line of curvature).

Let a', b' be the semi-axes of the central section of the ellipsoid by a plane parallel to the tangent plane at Y; then, since PY and UY meet at right angles at Y, we have

$$\frac{1}{D^2} + \frac{1}{D'^2} = \frac{1}{a'^2} + \frac{1}{b'^2} = \frac{a'^2 + b'^2}{a'^2 b'^2};$$

* Along a line of curvature pD is constant $= \mu$, also the other axis of the central section is constant $= \dfrac{abc}{pD} = \lambda$, the constant which enters into the second equation. The particular line of curvature will therefore be known by the particular value of λ or μ.

therefore
$$\frac{1}{p^2D^2}+\frac{1}{p^2D'^2}=\frac{a'^2+b'^2}{p^2a'^2b'^2};$$

therefore
$$\frac{1}{a^2c^2}+\frac{1}{\mu^2}=\frac{a^2+b^2+c^2-r^2}{a^2b^2c^2},$$

where r is the distance of Y from the centre; therefore

$$r^2=a^2+c^2-\frac{a^2b^2c^2}{\mu^2}=a^2+c^2-\lambda^2.$$

Substituting this value of r in (I), we get for the equation to the projection of this sphero-conic

$$x^2+y^2=\frac{b^2}{b^2-c^2}(a^2-\lambda^2),$$

and this is the equation to the circle which is the locus of the foot of the perpendicular, from the focus, on the tangent to the projection of the line of curvature (II).

PROP. 2. *The locus of the point of intersection of geodesic tangents to a line of curvature, at right angles, is the sphero-conic whose projection is the circle which is the locus of the point of intersection of tangents at right angles, to the projection of the line of curvature.*

Let PT, QT be the geodesic tangents, then along both of them $pD=\mu$, and therefore, at T, D is the same for both, and since they are at right angles

$$\frac{2}{D^2}=\frac{1}{a'^2}+\frac{1}{b'^2};$$

and therefore
$$\frac{2}{\mu^2}=\frac{a^2+b^2+c^2-r^2}{a^2b^2c^2};$$

therefore
$$r^2=a^2+b^2+c^2-\frac{2a^2b^2c^2}{\mu^2}$$
$$=a^2+b^2+c^2-2\lambda^2.$$

Substituting this value of r in (I), we get for the equation to the projection

$$x^2+y^2=\frac{b^2}{b^2-c^2}\{(a^2-\lambda^2)+(b^2-\lambda^2)\},$$

and this is the equation to the circle which is the locus of the point of intersection of tangents at right angles to (II).

PROP. 3. *If*, in Prop. 2, *we draw one tangent to a line of curvature* λ, *and the other to a line of curvature* λ', *the locus of T will be a sphero-conic whose projection is the circle which is the locus of the point of intersection of tangents at right angles, the one drawn to the projection of* λ, *the other to that of* λ'.

For, proceeding as before, we have

$$\frac{1}{\mu^2} + \frac{1}{\mu'^2} = \frac{a^2 + b^2 + c^2 - r^2}{a^2b^2c^2};$$

and therefore $\quad r^2 = a^2 + b^2 + c^2 - \lambda^2 - \lambda'^2,$

and substituting in (I), the equation to the projection becomes

$$x^2 + y^2 = \frac{b^2}{b^2 - c^2}(a^2 + b^2 - \lambda^2 - \lambda'^2),$$

the locus required.

Since writing Props. 1, 2, and 3 I found that they were particular cases of more general propositions, which I proceed to prove.

(I) *If from any point of an ellipsoid two geodesic tangents be drawn to a line of curvature, the angle between them is equal to the angle between the tangents from the projection of the point on the circular section, to the projection of the line of curvature; the projections being by lines parallel to the least axis of the ellipsoid.*

Let a', b' be the semiaxes of the diametral section of the point of intersection of the geodesic tangents, 2β the angle between them; then, with the same notation as before,

$$\frac{1}{D^2} = \frac{\sin^2\beta}{a'^2} + \frac{\cos^2\beta}{b'^2};$$

and therefore $\quad a'^2\cos^2\beta + b'^2\sin^2\beta = \dfrac{a^2b^2c^2}{\mu^2} = \lambda^2;$

therefore $$\tan^2\beta = -\frac{a'^2 - \lambda^2}{b'^2 - \lambda^2},$$

and therefore $\quad \tan 2\beta = 2\dfrac{\{(a'^2 + b'^2)\lambda^2 - a'^2b'^2 - \lambda^4\}^{\frac{1}{2}}}{a'^2 + b'^2 - 2\lambda^2} \; \ldots\ldots \; (1).$

Now $\quad a'^2 + b'^2 = a^2 + b^2 + c^2 - r^2,$

$$a'^2b'^2 = \frac{a^2b^2c^2}{p^2},$$

and
$$r^2 = x^2 + y^2 + z^2$$
$$= c^2 + x^2 . \frac{a^2 - c^2}{a^2} + y^2 . \frac{b^2 - c^2}{b^2}$$
$$= c^2 + (X^2 + Y^2) \frac{b^2 - c^2}{b^2},$$

where X, Y are the coordinates of the projection on the circular section; and

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$$
$$= \frac{1}{c^2} \left\{1 - \frac{b^2 - c^2}{b^2} \left(\frac{X^2}{a^2} + \frac{Y^2}{b^2}\right)\right\};$$

therefore
$$a'^2 + b'^2 = a^2 + b^2 - \frac{b^2 - c^2}{b^2} (X^2 + Y^2),$$
$$a'^2 b'^2 = a^2 b^2 \left\{1 - \frac{b^2 - c^2}{b^2} \left(\frac{X^2}{a^2} + \frac{Y^2}{b^2}\right)\right\}.$$

Substituting in (1) and reducing, we get

$\tan 2\beta$
$$= \frac{2\sqrt{\left\{X^2 \frac{b^2}{b^2 - c^2}(b^2 - \lambda^2) + Y^2 \frac{b^2}{b^2 - c^2}(a^2 - \lambda^2) - \frac{b^4}{(b^2 - c^2)^2}(a^2 - \lambda^2)(b^2 - \lambda^2)\right\}}}{\left\{X^2 + Y^2 - \frac{b^2}{b^2 - c^2}(a^2 - \lambda^2 + b^2 - \lambda^2)\right\}};$$

and therefore the angle between the tangents from the point XY, to the projection of the line of curvature

$$\frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = \frac{b^2}{b^2 - c^2}$$

is equal to 2β (Salmon's *Conic Sections*, p. 151).

(II) Since the umbilici U, V are the points in which the umbilical focal conic cuts the ellipsoid, and the umbilical focal conic is the limit of a confocal hyperboloid, it follows that the umbilici may be regarded as a line of curvature. Hence if any point T on the ellipsoid be joined to U and V by geodesic lines, and if T', S, H be the projections of T, U, V on the circular section, then the angle UTV is equal to the angle $ST'H$.

(III) If from T we draw a geodesic line touching a line of curvature in P, and join UT, then the angle UTP is equal to the angle between ST' and the tangent $T'P'$, to the projection of the line of curvature. For if TQ be the other geodesic tangent to the line of curvature, from T, and $T'Q'$ the other tangent to its projection, then

$$\angle P'T'Q' = \angle PTQ, \text{ and } \angle ST'H = \angle UTV;$$

therefore $$2\angle ST'P' = 2\angle UTP,$$

and therefore $$\angle ST'P' = \angle UTP.$$

(IV) If from T we draw a geodesic line TP touching one line of curvature, and TQ touching another line of curvature; and from T' draw tangents $T'P'$, $T'Q'$ to the projections of the lines of curvature, then $\angle PTQ = \angle P'T'Q'$. For join UT, ST', then, by (III),

$$\angle UTP = \angle ST'P',$$

and $$\angle UTQ = \angle ST'Q';$$

therefore $$\angle PTQ = \angle P'T'Q'.$$

It will be seen that Props. 1, 2, and 3 follow at once from (I), (III), and (IV).

From (II) it follows that if the angle UTV be constant, the locus of T projects into the circle which is the locus of T' when $ST'H$ is equal to the same constant.

Again, if from U a geodesic line UT be drawn making a constant angle with any tangent TP to a line of curvature, the locus of T projects into a circle. For if from S be drawn ST' making the same constant angle with any tangent $T'P'$ to the projection of the line of curvature, we know that the locus of T' is a circle (Salmon's *Conics*).

Many other such deductions might evidently be made.

August, 1862.

THE EQUATIONS OF THE STRAIGHT LINE AND PLANE IN TRILINEAR AND QUADRIPLANAR COORDINATES.

By W. Esson, M.A., Fellow and Tutor of Merton College, Oxford.

I.—*The Straight Line.*

Lemma. Three parallel right lines drawn from the angles A, B, C of a triangle are intersected by any right line in points L, M, N. The parallelograms formed by AL, MN; BM, NL; CN, LM are together equal to twice the area of the triangle ABC.

This proposition is well known and capable of very simple proof.

1. Let AL, BM, CN be perpendicular to LMN, and let their values be p, q, r; and let α, β, γ be the trilinear coordinates of any point P in LMN. Twice the area of BCP is, by the Lemma, equal to $qNP+rPM$, hence

$$a\alpha = qNP + rPM,$$
$$b\beta = rLP + pPN,$$
$$c\gamma = pMP + qPL;$$

therefore $$pa\alpha + qb\beta + rc\gamma = 0 \text{ .... (1)},$$

the equation of a right line in trilinear coordinates.

2. Let α' be the perpendicular from any point on (1). The equation of a line parallel to (1) through the point is plainly

$$(p-\alpha')\,a\alpha + (q-\alpha')\,b\beta + (r-\alpha')\,c\gamma = 0;$$

therefore $$2\Delta\alpha' = pa\alpha + qb\beta + rc\gamma.$$

Hence if $$\frac{pa}{2\Delta} = l, \quad \frac{qb}{2\Delta} = m, \quad \frac{rc}{2\Delta} = n,$$

$$\alpha' = l\alpha + m\beta + n\gamma \text{ .... (2)}.$$

In what follows lmn will be restricted to these values.

3. Let θ, ϕ, ψ be the angles between the right line and the sides of the triangle. Draw a perpendicular from A on

BC and parallels to it through B, C; let these parallels meet the line in $L'M'N'$, then, by Lemma,

$$AL',\ M'N' + BM',\ N'L' + CN',\ L'M' = 2\Delta,$$

but $\quad AL' \cos\theta = p, \quad BM' \cos\theta = q, \quad CN' \cos\theta = r,$

and the altitude of the parallelogram AL', $M'N'$; is a, of

$$BM',\ N'L' - b\cos C, \text{ and of } CN',\ L'M' - c\cos B,$$

hence $\quad 2\Delta\cos\theta = pa - qb\cos C - rc\cos B,$

$$\left.\begin{aligned} \text{or} \quad & \cos\theta = l - m\cos C - n\cos B. \\ \text{Similarly} \quad & \cos\phi = m - n\cos A - l\cos C \\ & \cos\psi = n - l\cos B - m\cos A \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

But $\quad AL.MN + BM.NL + CN.LM = 2\Delta,$

or $\quad pa\cos\theta + qb\cos\phi + rc\cos\psi = 2\Delta;$

therefore $\quad l\cos\theta + m\cos\phi + n\cos\psi = 1 \ldots\ldots\ldots\ldots (4),$

hence from (3) and (4)

$$l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2ln\cos C = 1 \ldots (5),$$

which may be written shortly

$$\Sigma l^2 - 2\Sigma mn\cos A = 1.$$

4. The reduction of any equation

$$L\alpha + M\beta + N\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

to the form $l\alpha + m\beta + n\gamma$ is effected by dividing it by

$$(\Sigma L^2 - 2\Sigma MN\cos A)^{\frac{1}{2}}.$$

This follows at once from (5).

Hence the perpendicular from any point on (6) is

$$\alpha' = \frac{L\alpha + M\beta + N\gamma}{(\Sigma L^2 - 2\Sigma MN\cos A)^{\frac{1}{2}}} \ldots\ldots\ldots\ldots\ldots (7)$$

5. The angle between two given straight lines is easily found in the usual form from equations (3).

6. Let L_1, M_1, N_1, &c. be the minors of the determinant $|\alpha_1\beta_2\gamma_3|$. Then the equations of the sides of a triangle ($A'B'C'$, $a'b'c'$, Δ') may be written (by (7))

$$\left.\begin{aligned} L_1\alpha + M_1\beta + N_1\gamma &= Q_1\alpha' \\ L_2\alpha + M_2\beta + N_2\gamma &= Q_2\beta' \\ L_3\alpha + M_3\beta + N_3\gamma &= Q_3\gamma' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (8),$$

where α', β', γ' are the coordinates of any point with reference to $A'B'C'$,

$$Q_1 = (\Sigma\,|\beta_2\gamma_3|^2 - 2\Sigma\,|\gamma_2\alpha_3|\,|\alpha_2\beta_3|\cos A)^{\frac{1}{2}},$$

and Q_2, Q_3 similar quantities.

Solving equations (8), we have

$$|\alpha_1\beta_2\gamma_3|\,\alpha = \alpha_1 Q_1\alpha' + \alpha_2 Q_2\beta' + \alpha_3 Q_3\gamma' \ldots\ldots\ldots (9),$$

but since α_1, α_2, α_3 are the perpendiculars from $A'B'C'$ upon α, we have, by 2,

$$2\Delta'.\alpha = \alpha_1 a'\alpha' + \alpha_2 b'\beta' + \alpha_3 c'\gamma';$$

therefore
$$\frac{2\Delta'}{|\alpha_1\beta_2\gamma_3|} = \frac{a'}{Q_1} = \frac{b'}{Q_2} = \frac{c'}{Q_3},$$

and since a' depends only on $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$, and Q_1 contains only these coordinates, each of these ratios is equal to a constant k suppose; therefore

$$2\Delta' = k\,|\alpha_1\beta_2\gamma_3|,$$

let
$$\Delta' = \Delta;$$

then
$$2\Delta = k\left|\frac{2\Delta}{a}, \frac{2\Delta}{b}, \frac{2\Delta}{c}\right|;$$

$$k = \frac{abc}{(2\Delta)^2};$$

therefore
$$2\Delta' = \frac{abc}{(2\Delta)^2}\,|\alpha_1\beta_2\gamma_3|,$$

and
$$a' = \frac{abc}{(2\Delta)^2}\,\{\Sigma\,|\beta_2\gamma_3|^2 - 2\Sigma\,|\gamma_2\alpha_3|\,|\alpha_2\beta_3|\cos A\}^{\frac{1}{2}},$$

which are expressions for the area of a triangle in terms of the coordinates of its angular points; and for the length of a line in terms of the coordinates of its extremities.

II.—*The Plane.*

Lemma. Three parallel right lines drawn from the angles A, B, C, D of a tetrahedron are intersected by any plane in points K, L, M, N. The prisms formed by AK, LMN; BL, MNK; CM, NKL; DN, KLM are together equal to three times the volume of the tetrahedron.

This is capable of very simple proof.

1. Let AK, BL, CM, DN be perpendicular to $KLMN$, and let their values be p, q, r, s, and let $\alpha, \beta, \gamma, \delta$ be the quadriplanar coordinates of any point P in $KLMN$. Thrice the volume of $BCDP$ is, by the Lemma, equal to

$$qMNP + rNPL + sPLM,$$

hence if $abcd$ are the areas of the planes opposite $ABCD$

$$a\alpha = qMNP + rNPL + sPLM,$$

$$b\beta = rNKP + sKPM + pPMN,$$

$$c\gamma = sKLP + pLPN + qPNK,$$

$$d\delta = pLMP + qMPK + rPKL;$$

therefore $$pa\alpha + qb\beta + rc\gamma + sd\delta = 0,$$

the equation of the plane in quadriplanar coordinates. The convention of signs with regard to triangles is easily seen.

2. By a similar proof to I. (2)

$$\alpha' = k\alpha + l\beta + m\gamma + n\delta,$$

where $$\frac{k}{pa} = \frac{l}{qb} = \frac{m}{rc} = \frac{n}{sd} = \frac{1}{3V},$$

V being the volume of the tetrahedron of reference.

3. A similar proof to I. (3) leads us at once to

$$\cos\theta = k - l\cos\widehat{ab} - m\cos\widehat{ac} - n\cos\widehat{ad},$$

(where θ is the angle between the plane and the face BCD), and also to

$$\Sigma k^2 - 2\Sigma kl\cos\widehat{ab} = 1.$$

4. Hence if the equation of any plane is

$$K\alpha + L\beta + M\gamma + N\delta = 0,$$

the perpendicular on it from any point is

$$\frac{K\alpha + L\beta + M\gamma + N\delta}{(\Sigma K^2 - 2\Sigma KL\cos\widehat{ab})^{\frac{1}{2}}}.$$

5. The condition that $(klmn)$ should be perpendicular to $(k'l'm'n')$ is the same as the condition that the right line

$$\frac{\alpha-\alpha'}{\cos\theta}=\frac{\beta-\beta'}{\cos\phi}=\frac{\gamma-\gamma'}{\cos\psi}=\frac{\delta-\delta'}{\cos\omega}=r$$

should lie in the plane $(k'l'm'n')$ which is plainly

$$k'\cos\theta+l'\cos\phi+m'\cos\psi+n'\cos\omega=0,$$

or $$\Sigma kk'-\Sigma(kl'+k'l)\cos\widehat{ab}=0;$$

and the left-hand member of this equation is the cosine of the angle between the two planes.

6. By a process analogous to I. (6) we arrive at the following formulæ for the volume of any tetrahedron and triangle in terms of the coordinates of their angular points

$$3V'=\frac{abcd}{(3V)^3}\,|\,\alpha_1\beta_2\gamma_3\delta_4\,|,$$

$$a'=\frac{abcd}{(3V)^3}\,\{\Sigma\,|\,\beta_2\gamma_3\delta_4\,|^2-2\Sigma\,|\,\beta_2\gamma_3\delta_4\,|\,|\,\gamma_2\delta_3\alpha_4\,|\cos\widehat{ab}\}^{\frac{1}{2}}.$$

ON THE WALKING AND GRAZING OF QUADRUPEDS.

By William Walton, M.A., Trinity College.

Borelli, in his very interesting work on Animal Mechanics, entitled *De Motu Animalium*, Pars Prima, p. 263, published in Rome in the year 1680, discusses the question of the walking of quadrupeds, pointing out the erroneous conceptions which had prevailed on this subject among philosophers, artists, and sculptors. The ordinary idea appears to have been that, in walking, horses and other quadrupeds proceed by the diagonal action of the legs, the right fore leg and left hind leg being raised simultaneously, then the other two diagonal legs, and so on: "Talis porrò erronea imaginatio adeò invaluit, ut in statuis equestribus æneis, et

marmoreis antiquis, et recentibus, semper duo pedes è diametro oppositi à Terra suspensi exculpti, et in tabulis depicti sint."

Borelli properly observes that not only is such a notion contrary to fact, but also that it would be incompatible with the stability of a horse in motion: he says, in fact, A, B, C, D, being the feet of a horse, "quando simul tempore elevantur, et moventur duo pedes diametraliter oppositi, B, D, pariter tota moles animalis inniti debet super duos pedes firmos, scilicet linea propensionis insistet perpendiculariter non super spatium amplum, sed super lineam AC; ergò pariter animal vacillabit, et proinde infirmam et instabilem posituram tunc temporis habebit."

My primary object in this communication to the *Quarterly Journal* is to throw light upon the mechanics of grazing. I shall however commence with the consideration of the movement of walking, partly because Borelli has somewhat obscured his substantially good explanation of walking by the introduction of a rectangular instead of a parallelogrammic posture of the feet, and partly for the sake of contrasting the mechanics of these two movements.

The postures of the feet in walking are represented in the figs. 45, 46, 47, 48, fig. 49 being the repetition of fig. 45 and representing the commencement of the successive series of postures.

In each of these figures A and D are the left feet, B and C the right feet, A and B the fore feet, C and D the hind feet. These figures are of course not separate in actual walking, a partial overlapping taking place between the areas of successive postures. The position of the centre of gravity of the quadruped, orthogonally projected on the ground, is indicated by the letter G, which is equidistant from the lines AD, BC, and of which the relation to the diagonals in each figure will be observed. The motion of G is rectilinear, advancing, at each step, through a space equal to half the projection of AB or CD upon AD or BC.

In the transformation of the posture (45) into the posture (46), the strong propelling power of the hind leg C securely transfers G, from a position of semi-stability in the diagonal BD, into the area of the triangle ACD. During the transformation of the posture (46) into the posture (47), G lies within the area of the triangle ACD and accordingly, the stability in virtue of the three stationary legs A, C, D, being perfect, the smaller propelling power of the fore leg B is adequate to its duty, even when the animal is agitated

by the accidental disturbances of its journey. The postures (47) and (48) are merely the postures (45) and (46) inverted, the posture (49) being the first posture of the new series: the mechanics of the subsequent progress are therefore the same as before. It is important to observe that in walking the successive foot-falls succeed each other at approximately equal intervals: in virtue of this fact the centre of gravity of the quadruped advances at a uniform pace. Moreover the back of the quadruped, in walking, is approximately horizontal, his right pair of legs and his left pair being each in a vertical plane. When the animal is employed in traction, his legs incline considerably forward; not much so when he is walking at his own discretion.

I proceed now to offer a few remarks on the peculiarities of grazing as distinguished from walking.

The successive postures of the feet, excepting a slight modification which I will afterwards mention, are substantially the same as in walking. The postures (46) and (48) however are attitudes of considerable duration, while the postures (45) and (47) are almost merely transitional. In consequence of these facts, a person looking at grazing sheep or cattle, would be struck by the appearance of a diagonal action of the legs.

In order to allow the back of the quadruped to lean downwards from the tail towards the head, an arrangement by which the muzzle is brought near to the grass, the two fore legs A and B, in the postures (46) and (48), assume a straddled attitude, an attitude convenient also for ensuring stability: moreover the leg A in posture (46) and the leg B in posture (48) is more advanced than in the postures of walking: in the posture (46), the principal stress is exerted on the legs A, C, more especially on A, while the actions of the legs B, D, undertake more especially the duty of rocking the animal to and fro about the diagonal AC, thereby affording greater mobility to the muzzle, the mean position of which, it may be observed, is nearer to the foot A than to the foot B. The position of G in the posture (46) relieves the leg B of some stress, and thereby enables it the more easily to move into its position in the posture (47). Observations analogous to those which I have made respecting the posture (46) are of course applicable to the posture (48). It may be observed that the peculiar straightness of the fore legs of a quadruped renders them able to sustain well the great longitudinal stress to which they are alternately subjected in grazing.

From what has been said above, it is evident that the figs. (46) and (48), which represent the postures of walking, must be somewhat modified to accommodate them to grazing postures: in fact the straddling of the fore legs effects a certain elongation of the side AB of the figure, the diagonal CA in (46) and DB in (48) being somewhat elongated by reason of the more advanced attitude of A, B, respectively.

The postures (45) and (47) are, as I have already remarked, little more than transitional: as a reason for this it may be stated that, in these positions, the quadruped is at full stretch in the directions of both diagonals, and that accordingly he has less mobility in grazing.

Horses, miserable and frail from age and hard usage, or ignorant of the manners of the field from constant life in stables, and young colts shaky in their legs, may sometimes be seen grazing with their feet in the abnormal posture, variously modified, which is represented in fig. (50), the fore legs straddling enormously. In the mechanical support of the animal, the four legs take a more nearly equal share than in correct grazing, and accordingly the strength of no one leg is severely tasked: the process of grazing is however under these circumstances essentially intermittent, whereas the behaviour of normal grazing renders it continuous.

Should any man wish to verify these mechanical actions of the legs in grazing by an appeal to his own actual sensations, he may easily do so by going on all fours like a sheep at grass, and imitating the grazing attitudes.

July 20, 1861.

ON SOME FORMULÆ RELATING TO THE DISTANCES OF A POINT FROM THE VERTICES OF A TRIANGLE, AND TO THE PROBLEM OF TACTIONS.

By A. Cayley.

The relation between the distances of four points 1, 2, 3, 4 in a plane is

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

where, see my paper "Note on the value of certain Determinants the terms of which are the squared distances of Points in a plane or in space," *Quarterly Journal of Mathematics*, t. III., p. 275 (1859), the determinant is

$$= \Sigma\overline{12}^2.\overline{23}^2.\overline{34}^2 - \Sigma\overline{12}^2.\overline{34}^2.\overline{43}^2 - \Sigma\overline{12}^2.\overline{23}^2.\overline{31}^2,$$

an identity which subsists without the aid of the relations $12 = 21$, &c., and in which the Σ, Σ, Σ contain 24, 12, and 8 terms respectively.

Writing $23 = f$, $31 = g$, $12 = h$, $14 = a$, $24 = b$, $34 = c$, the determinant is

$$\begin{aligned} = 2\,\{&g^2h^2(b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2) \\ &+ b^2c^2(g^2+h^2) + c^2a^2(a^2+f^2) + a^2b^2(f^2+g^2) \\ &- a^2f^2(a^2+f^2) - b^2g^2(b^2+g^2) - c^2h^2(c^2+h^2) \\ &- b^2c^2f^2 - c^2a^2g^2 - a^2b^2h^2 \qquad\qquad \} \\ = -2\,\square&, \end{aligned}$$

if $\square$ denote the function in { } with the signs reversed. The function $\square$ may be expressed in the form

$$\begin{aligned} \square = {} & a^4f^2 + b^4g^2 + c^4h^2 + f^2h^2g^2 \\ &+ (a^2f^2 + b^2c^2)(f^2 - g^2 - h^2) \\ &+ (b^2g^2 + c^2a^2)(g^2 - h^2 - f^2) \\ &+ (c^2h^2 + a^2b^2)(h^2 - f^2 - g^2), \end{aligned}$$

and also in the form

$$\square = U^2 + (f+g+h)\,V,$$

if for shortness

$$U = a^2f + b^2g + c^2h + fgh,$$

$$\begin{aligned} V = &(a^2f^2 + b^2c^2)(f-g-h) \\ &+ (b^2g^2 + c^2a^2)(g-h-f) \\ &+ (c^2h^2 + a^2b^2)(h-f-g)\,; \end{aligned}$$

and it may be remarked that since $\square$ is an even function of f, g, h, we may in this last formula change at pleasure the signs of these quantities; we thus obtain in all four similar forms of the function $\square$.

It is clear that considering a triangle, and any point in the plane of the triangle, f, g, h may be taken to denote the sides of the triangle, and a, b, c the distances of the point from the vertices. And the equation $\square = 0$ is the relation connecting the sides and distances.

The equation $f+g+h=0$ denotes that the vertices are *in lineo*, and when this equation is satisfied we have

$$U = a^2f + b^2g + c^2h + fgh = 0,$$

which is in fact, as it is easy to see, the relation connecting the distances of a point from any three points *in lineo*.

For a, b, c write $a+x$, $b+x$, $c+x$; x will be the radius of a circle touching the circles, radii a, b, c, described about the vertices as centres. The equation $\square = 0$ becomes after all reductions

$$\begin{aligned} &U^2 - (f+g+h)\,V \\ &+ x[4\,U(af+bg+ch) \\ &\quad - 2\,(f+g+h)\,\{(af^2 + bc\,(b+c))(f-g-h) \\ &\qquad\qquad + (bg^2 + ca\,(c+a))(g-h-f) \\ &\qquad\qquad + (ch^2 + ab\,(a+b))(h-f-g)\}] \\ &+ x^2\,[f^2\,\{-4a^2 + 6a\,(b+c) - 6bc\} \\ &\quad + g^2\,\{-4b^2 + 6b\,(c+a) - 6ca\} \\ &\quad + h^2\,\{-4c^2 + 6c\,(a+b) - 6ab\}] = 0, \end{aligned}$$

which is a quadratic equation only: the two circles thus obtained are those which touch the given circles all three externally or all three internally. But by changing in every possible manner the signs of a, b, c we obtain in all four

equations giving the eight tangent circles. It may be noticed that if as before $f+g+h=0$, $U=0$, then not only the constant term vanishes, but the coefficient of x also vanishes or the equation becomes simply $x^2=0$.

In particular, suppose $f=b+c$, $g=c+a$, $h=a+b$; developing this *de novo*, and putting for shortness

$$\begin{aligned} a+b+c &= p, \\ bc+ca+ab &= q, \\ abc &= r, \end{aligned}$$

we find

$$U = 2\,(px^2+2qx+pq-2r),$$

$$V = 2\,\{px^4+4qx^3+(2pq+12r)\,x^2+4q^2x+pq^2-4qr\},$$

and then the equation $\square = U^2-2pV=0$ gives

$$\begin{aligned} \tfrac{1}{4}\square &= (px^2+2qx+pq-2r)^2 \\ &\qquad -p\,\{px^4+4qx^3+(2pq+12r)\,x^2+4q^2x+pq^2-4qr\} \\ &= 4\,\{(q^2-4pr)\,x^2-2qrx+r^2\}, \end{aligned}$$

so that we have

$$\begin{aligned} \tfrac{1}{16}\square &= (q^2-4pr)\,x^2-2qrx+r^2 \\ &= (qx-r)^2-4prx=0, \end{aligned}$$

and thence

$$qx-r=\pm x\sqrt{(pr)}, \text{ or } x=\frac{r}{q\mp\sqrt{(pr)}},$$

which gives the radii of the circles inscribed in and circumscribed about the three circles radii a, b, c, whereof each touches the two others: a formula given by Descartes, *Opera*, *Franc.* 1792, *Epist.*, *Pars* III., p. 261, in a letter to the Princess Elisabeth, viz. Descartes has

$$\begin{aligned} (d^2e^2+d^2f^2+e^2f^2-2def^2-2d^2ef-2de^2f)\,x^2 \\ -2\,(de^2f^2+d^2ef^2+d^2e^2f)\,x+d^2e^2f^2=0, \end{aligned}$$

which putting a, b, c for his d, e, f, becomes *ut suprà*

$$x^2\,(q^2-4pr)-2qrx+r^2=0.$$

In conclusion I notice the following formula which is obtained without difficulty, viz. if as before we have a triangle the sides whereof are f, g, h, and if a, b, c are the distances of a point from the vertices (so that as before $\square=0$)

then the perpendicular distances of the point from the sides, each perpendicular distance divided by the perpendicular distance of the opposite vertex from the same side, are as follows: viz. the quotient for the side f is

$$= \frac{1}{16\Delta^2}\left[(b^2-c^2)(g^2-h^2)+f^2(b^2+c^2+g^2+h^2-2a^2)-f^4\right],$$

where Δ is the area of the triangle. It is clear that we ought to have

$$\Sigma\left\{(b^2-c^2)(g^2-h^2)+f^2(b^2+c^2+g^2+h^2-2a^2)-f^4\right\}=16\Delta^2,$$

and this equation in fact reduces itself to

$$2g^2h^2+2h^2f^2+2f^2g^2-f^4-g^4-h^4=16\Delta^2,$$

which is right.

2, Stone Buildings, W.C.,
17*th September*, 1862.

THE END OF VOL. V.

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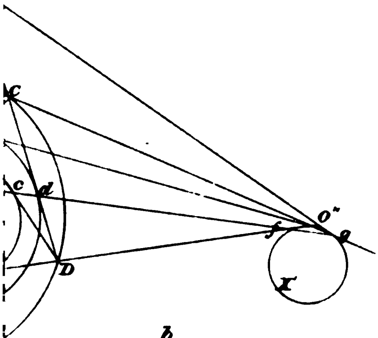
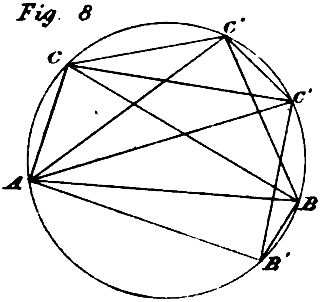
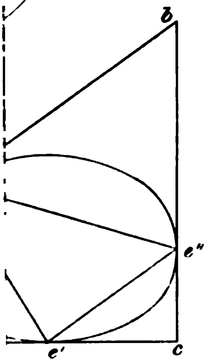


Fig. 8





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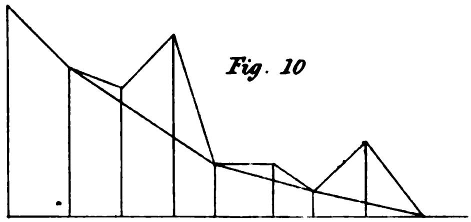

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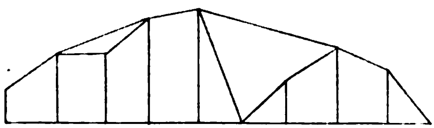

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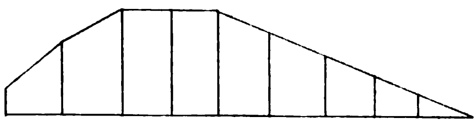


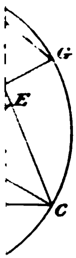


Fig. 15

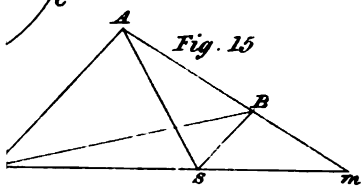
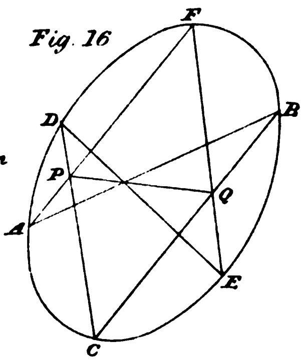


Fig. 16



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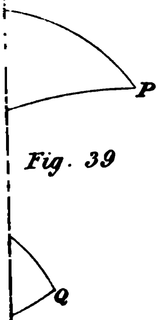


Fig. 39

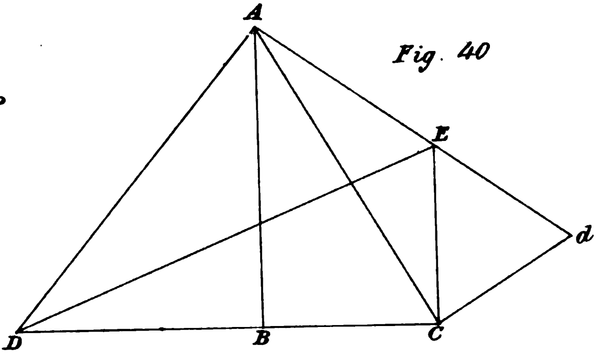
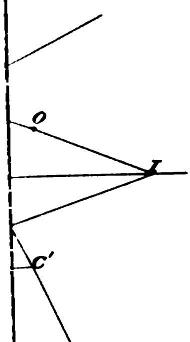


Fig. 40

Fig. 42.



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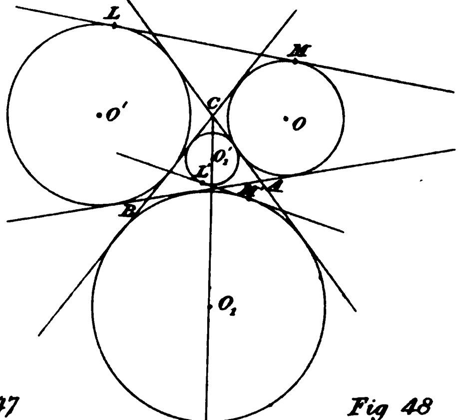

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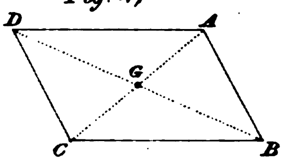
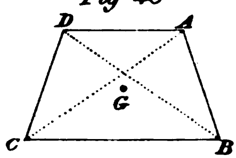


Fig 48



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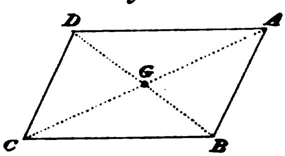
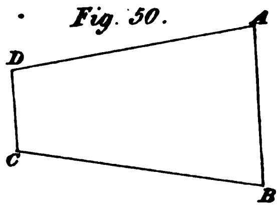


Fig. 50.



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